1. If M is the sum of submodules N and L is it true that $\mathrm{Ass}(M) = \mathrm{Ass}(N) \cup \mathrm{Ass}(L)$?

Answer: No. Take $A = \mathbf{Z}$, $M = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}$, $N = \mathbf{Z} \cdot (1, 1)$, and $L = \mathbf{Z} \cdot (0, 1)$. Then $\mathrm{Ass}(M) = \{2\mathbf{Z}, 0\}$, but each of N and L is isomorphic to Z and has $\mathrm{Ass} = \{0\}$.

2. Let $I = (x, y) \subset A = k[x, y, z]/(xy - z^2)$. Find $\mathrm{Ass}(A/I^2)$. (Optional: find a primary decomposition of I^2).

Answer: We have $(x, y, z) \subseteq \mathrm{rad}(I^2)$, since $x^2 = x^2 \in I^2, y^2 = y^2 \in I^2, z^2 = xy \in I^2$. Since (x, y, z) is maximal and $I^2 \neq (1)$, this means we have equality. Since (x, y, z) is maximal, we get that I^2 is (x, y, z)-primary, so its Ass is just $\{(x, y, z)\}$, and it is its own primary decomposition.

3. Let A be Noetherian and M, N finite modules over A. Show that $\mathrm{Ass}(\mathrm{Hom}(M, N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N)$.

Proof: Man, you guys really need to learn how to work with the commutative algebra instead of against it. It's a Tao thing, wei wu wei. Here are the lemmas which make the problem trivial (the first one could probably be used to shorten the exposition of all this Ass stuff):

Lemma 1: Let A be a ring, M an A-module. Then

$$\mathrm{Ass}(M) = \{P \in \mathrm{Spec}(A) \mid \mathrm{Hom}(A/P, M)_P \neq 0\}.$$

Proof: We have $\mathrm{Hom}(A/P, M)_P \neq 0$ if and only if there is a $\phi \in \mathrm{Hom}(A/P, M)$ with $\mathrm{Ann}(\phi) \subseteq P$. But giving a $\phi \in \mathrm{Hom}(A/P, M)$ is exactly the same as giving an element (namely $\phi(1)$) of M annihilated by P; and the annihilator of ϕ is then the same as the annihilator of that element; so the condition on this element is that it have annihilator both contained in and containing P. $\square$

Lemma 2: Let A be a ring, B a flat A-algebra (e.g. $B = A_P$ for some prime P), M a finitely presented A-module (this is equivalent to finitely generated if A is Noetherian), and N an arbitrary A-module. Then there's a natural isomorphism of B-modules

$$\mathrm{Hom}_A(M, N)_B \xrightarrow{\sim} \mathrm{Hom}_B(M_B, N_B).$$

Proof: First we give the map: send $\phi \otimes b$ to $m \otimes b' \mapsto \phi(m) \otimes (bb')$. This is clearly natural, well-defined, B-linear, etc. etc.

Now we check that it's an isomorphism in the special case $M = A^n$ for some $n \geq 0$. Well, in that case, $\mathrm{Hom}_A(M, N) \cong N^n$, and $\mathrm{Hom}_B(M_B, N_B) \cong (N_B)^n$, and it's easy to see that via these identifications, the above map becomes the obvious map $(N^n)_B \to (N_B)^n$, which we know to be an isomorphism.

For the general case, since M is finitely-presented, we have an exact sequence

$$A^m \to A^n \to M \to 0.$$

To this we can apply the functors $\mathrm{Hom}_A(-, N)_B$ and $\mathrm{Hom}_B((-)_B, N_B)$, both of which are contravariant left-exact by general nonsense and the flatness hypothesis;

then we get a commutative diagram where the rows are exact, the vertical arrows all instances of the above-defined map:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(M,N)_B & \longrightarrow & \mathrm{Hom}_A(A^n,N)_B & \longrightarrow & \mathrm{Hom}_A(A^m,N)_B \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_B(M_B,N_B) & \longrightarrow & \mathrm{Hom}_B((A^n)_B,N_B) & \longrightarrow & \mathrm{Hom}_B((A^m)_B,N_B). \end{array}$$

Then by the first thing we checked, the last two vertical maps are isomorphisms; by diagram chasing, then, so is the first one, as desired. □

OK, armed with our lemmas, let's just compute. To find $\mathrm{Ass}(\mathrm{Hom}(M,N))$, by the first lemma, we should consider

$$\begin{aligned} \mathrm{Hom}(A/P, \mathrm{Hom}(M,N))_P &\cong \mathrm{Hom}(M, \mathrm{Hom}(A/P,N))_P \\ &\cong \mathrm{Hom}_{A_P}(M_P, \mathrm{Hom}(A/P,N)_P) \\ &\cong \mathrm{Hom}_k(M_P \otimes_{A_P} k, \mathrm{Hom}(A/P,N)_P). \end{aligned}$$

Here $k = A_P/PA_P$, the first isomorphism is because both sides are just bilinear maps $A/P \times M \to N$ localized at P, the second isomorphism is the second lemma, and the third isomorphism is because $\mathrm{Hom}(A/P,N)_P$ is killed by P, and is thus a k-module. But then look at what we have: we're working over a field, so it's trivial that this last thing is 0 if and only if $M_P \otimes_{A_P} k = 0$ or $\mathrm{Hom}(A/P,N)_P = 0$. By Nakayama (again we use M finite), this is the same as $M_P = 0$ or $\mathrm{Hom}(A/P,N)_P = 0$. By the first lemma, we're done. We see that the statement holds for general A, finitely presented M, and arbitrary N. □

3.16. Let B be a flat A-algebra. TFAE:

(1) $I^{ce} = I$ for all ideals $I \subseteq A$.

(2) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

(3) For every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}^e \neq (1)$.

(4) If $M \neq 0$ is an A-module, then $M_B \neq 0$.

(5) For every A-module M, the natural $M \to M_B$ is injective.

Proof: (1) ⇒ (2) is one of the problems on the last set.

(2) ⇒ (3): Say $\mathfrak{m} = P^c$ for $P \subseteq B$ prime. Then $\mathfrak{m}^e = P^{ce} \subseteq P \subsetneq (1)$.

(3) ⇒ (4): Let $m \in M$ with $m \neq 0$. The map $A \to M$ given by $a \mapsto a \cdot m$ has kernel $I = \mathrm{Ann}(m) \neq (1)$, so we get an exact

$$0 \to A/I \to M.$$

Tensoring gives, by flatness and one of our first exercises,

$$0 \to B/IB \to M_B.$$

But if $I \subseteq \mathfrak{m}$, then $IB \subseteq \mathfrak{m}^e \subsetneq (1)$, so $B/IB \neq 0$, so $M_B \neq 0$.

(4) $\Rightarrow$ (5): Let N be the kernel, so

$$0 \to N \to M \to M_B.$$

Flatness gives

$$0 \to N_B \to M_B \to M_B \otimes_A B.$$

However, this last map has a left inverse, namely $(m \otimes b) \otimes b' \mapsto m \otimes (bb')$, which means it's injective, so $N_B = 0$, so $N = 0$, as desired.

(5) $\Rightarrow$ (1): Take $M = A/I$; we get that $A/I \to B/IB$ is injective, which exactly says $I = I^{ce}$.

4.2. Let I be a radical ideal in a Noetherian ring A. Then I has no embedded primes.

Proof: Recall Spec(A/I) is noetherian, so it's a union of finitely many irreducible components, which correspond to minimal primes of A/I. Radical ideals corresponding to closed subsets of Spec, we get that I is the intersection of the (finitely many) primes minimal over I. This is a primary decomposition, and by minimality none are embedded.