1) Let M be an A-module. Prove that the following are equivalent:

(1) M is projective.
(2) M is a direct summand of a free module.
(3) For any surjection $N \twoheadrightarrow L$, the natural map $Hom(M, N) \to Hom(M, L)$ is surjective.
(4) Every surjection $\alpha : N \twoheadrightarrow M$ splits.

Proof: (3) is just a trivial restatement of (1). Now, (1) $\Rightarrow$ (4) since splitting is the same as lifting $id_M : M \to M$ to N. To show (4) $\Rightarrow$ (2), start with the map

$$\varphi : F = \oplus_{m \in M} A \to M$$

given by sending the m^{th} basis vector on the left to m. It's trivially surjective, so by hypothesis, it splits, say by $s : M \to F$. Then the map $M \oplus ker(\varphi) \to F$ given by $(m, x) \mapsto s(m) + x$ is an isomorphism, having inverse $f \mapsto (\varphi(f), f - s(\varphi(f)))$. I did something more general than this in section; remember split short exact sequences? Finally, we need (2) $\Rightarrow$ (1). We will show that whenever $M = \oplus_{i \in I} M_i$, then M is projective if and only if each M_i is. When we apply this to the case where all the M_i are A (which is trivially projective, since $Hom(A, N) \simeq N$ naturally), we deduce that all free modules are projective; then applying it again, we see direct summands of free modules are too. To prove the claim, recall the natural isomorphism $Hom(\oplus_{i \in I} M_i, N) \simeq \prod_{i \in I} Hom(M_i, N)$; this gives, for $N \twoheadrightarrow L$, a commutative diagram (the horizontal maps are the obvious ones, given by composition)

$$\begin{array}{ccc} Hom(\oplus_{i \in I} M_i, N) & \longrightarrow & Hom(\oplus_{i \in I} M_i, L) \\ \downarrow \sim & & \downarrow \sim \\ \prod_{i \in I} Hom(M_i N) & \longrightarrow & \prod_{i \in I} Hom(M_i, L), \end{array}$$

which shows that the top map is surjective if and only if the bottom one is; but the bottom one is if and only if each of its terms is, and this is exactly the desired statement.

2) Prove that a projective module is flat.

Proof: We'll use (2) above. Just as in the above problem, if we show that whenever $M = \oplus_{i \in I} M_i$, then M is flat if and only if each M_i is, then we will see that flatness of projective modules is equivalent to flatness of A, which is, dare I say, obvious. Well, let $N \hookrightarrow L$. We have a commutative diagram with vertical maps isomorphisms (expressing bilinearity of $\otimes$ over $\oplus$) and the horizontal maps the obvious ones:

$$\begin{array}{ccc} N \otimes (\oplus_{i \in I} M_i) & \longrightarrow & L \otimes (\oplus_{i \in I} M_i) \\ \downarrow \sim & & \downarrow \sim \\ \oplus_{i \in I} N \otimes M_i & \longrightarrow & \oplus_{i \in I} L \otimes M_i, \end{array}$$

so the top map is injective if and only if the bottom one is; the bottom one being injective trivially comes down to each M_i being flat.

3) Suppose $A = (A, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated projective A-module. Prove that M is free.

Proof: By Nakayama's lemma, there are $m_1, \ldots, m_n \in M$ which generate M and whose images in $M/\mathfrak{m}M$ form a basis (as an $A/\mathfrak{m}$-module). Define $\varphi : A^n \to M$ by sending the i^{th} basis vector to m_i; then this surjective, so by the argument of the first problem, there is an isomorphism $A^n \xrightarrow{\sim} M \oplus ker(\phi)$ given by $x \mapsto (\varphi(x), whatever)$. Reducing mod $\mathfrak{m}$, we get $k^n \xrightarrow{\sim} (M/\mathfrak{m}M) \oplus (ker(\phi)/\mathfrak{m}ker(\phi))$. But here we know the map is an isomorphism on the first factor, since it matches up the basis vectors; thus the second factor must be trivial. But $ker(\phi)$ is a submodule of A^n and is thus finitely generated, since A is noetherian; so Nakayama implies that $ker(\phi) = 0$, and hence ϕ is an isomorphism.

Actually, as many of your solutions showed, the noetherian hypothesis is unnecessary: since $A^n \xrightarrow{\sim} M \oplus ker(\phi)$, the module $ker(\phi) \simeq A^n/M$ is automatically finitely generated.

4) Suppose A is Noetherian and M is a finite A-module. Then M is projective if and only if M_P is a free A_P module for each prime $P \subseteq A$.

Proof: First suppose M is projective. Then it is the direct summand of a free module, say $M \oplus M' = F$; tensoring with A_P we see that M_P is also the direct summand of a free A_P-module, so it is projective. It's also finite, so by the previous problem it's free, as desired.

For the hard direction, I reserve the right to amuse myself by proving something stronger: we'll go through the intermediary statement that there exist $f_1, \ldots, f_n \in A$ with $Spec(A) = \cup_i D(f_i)$ such that M_{f_i} is a free A_{f_i}-module for each i. This is "locally free" in the true geometric sense. Remark, we will also only need that M_P is free for every *maximal* ideal P.

Lemma 1: Let M be a finite A-module (A Noetherian) and P a prime of A. If M_P is free, then so is M_f for some $f \notin P$.

Proof: Let $m_1/a_1, \ldots, m_n/a_n$ generate M_P, with the $m_i \in M$ and the $a_i \in A \backslash P$. Then the m_i define a map $\varphi : A^n \to M$ which is an isomorphism after localizing at P (the a_i are all units there); we want it to be an isomorphism after inverting some $f \in A \setminus P$. Let $K = ker(\varphi)$ and $C = coker(\varphi)$; since localizing is exact, we must have $K_P = 0$ and $C_P = 0$. Both K and C are also finite modules (the former since we're noetherian), so the lemma will follow if we can just show that if N is a finite A-module with $N_P = 0$, then $N_g = 0$ for some $g \notin P$ (for then we can apply this to K and C and take the product of the resulting g's). But $N_P = 0$ means that each generator is killed by something not in P; we can take the product of those elements and call it g, and that will kill the generators too, so $N_g = 0$, as desired. $\square$

This lemma shows us that, if M_P is free for each maximal M, there is a collection of elements f_i with M_{f_i} a free A_{f_i}-module and $Spec(A) = \cup_i D(f_i)$. By quasi-compactness, we can even assume the f_i are a finite set. Now,

Lemma 2: Let $A \to B$ be faithfully flat, and M a finite A-module (A noetherian). Then M is projective if and only if M_B is a projective (as a B-module, of course).

Proof: The "only if" follows from the same argument as the easy direction of this problem, and doesn't require anything to do with faithful flatness. For "if", recall I proved a lemma in the previous solutions which says that $Hom_A(M,N)_B$ is naturally isomorphic to $Hom_B(M_B, N_B)$ (given the hypotheses on M and A). If $N \to L$ is a surjection of A-modules, then $N_B \to L_B$ is a surjection of B-modules, so since M_B is projective, $Hom_B(M_B, N_B) \to Hom_B(M_B, L_B)$ is surjective, so $Hom_A(M,N)_B \to Hom_A(M,L)_B$ is surjective, so by faithful flatness, $Hom_A(M,N) \to Hom_A(M,L)$ is surjective, as desired. □

So, recall from the above that we have a finite cover $Spec(A) = \cup_i D(f_i)$ such that each M_{f_i} is a free A_{f_i}-module. I claim that the $B := \prod_i A_{f_i}$-module $\prod_i M_{f_i}$ is projective. Indeed, if $M_{f_i} \simeq A_{f_i}^{n_i}$, then $\prod_i M_{f_i} \simeq \prod_i A_{f_i}^{n_i}$ as B-modules; for n large enough it's clear how to make this a direct summand of $\prod_i A_{f_i}^n = B^n$: just fill out the deficient $A_{f_i}^{n_i}$'s. Now, we have the obvious map $A \to B$, and I claim that it's faithfully flat and that $\prod_i M_{f_i} = M_B$. If this is true, we'll be done by Lemma 2. The second claim is easy, since $B = \prod_i A_{f_i} = \oplus_i A_{f_i}$ and tensor is bilinear over $\oplus$. For the first claim (faithful flatness), we will check flatness, and that the map on Spec is surjective.

Flatness is again easy since B is a direct sum of flat modules (localizations are flat). To see faithful flatness, we just need this lemma:

Lemma 3: Let A_i be finitely many rings. Then $Spec(\prod_i A_i) = \coprod_i Spec(A_i)$ canonically; via this identification, if $\phi : A \to \prod_i A_i$ is any map of rings, say with i^{th} component ϕ_i, then the induced map of ϕ on Spec is the same as the map $\coprod_i Spec(A_i) \to Spec(A)$ induced by the maps ϕ_i separately.

Proof: Good exercise. The idea is that if P is a prime of A_i, then we get a prime of $\prod_i A_i$ by taking $\prod_j$ over A_j for $j \neq i$ and P for $j = i$, and this gives the desired identification. Every prime is of that form since a prime ideal of $\prod_i A_i$ can miss at most one of the "standard basis vectors" (idempotents, in this context), since the product of any two distinct of them is zero. The claim about the induced maps is pretty trivial once you have the correspondence. □

This lemma finishes the proof: it shows that the image of the map on *Spec* of $A \to B$ is $\cup_i D(f_i) = Spec(A)$.

For the problems about complexes, I'll use unbounded complexes, i.e. indexed by the integers instead of just the nonnegative integers. This is more general (you can just make all the negatively indexed modules 0 if you like) and just as easy. Then we also have homology groups in negative dimensions, a.k.a. cohomology groups.

I'll also call the ring R instead of A, so I can use A for other things.

4') Prove that a map $\alpha : C \to C'$ of complexes induces a map $\alpha_* : H_i(C) \to H_i(C')$ on homology for each i.

Proof: Well, the commutativity of the squares immediately implies that α sends $im(d_{i+1})$ to $im(d'_{i+1})$ and $ker(d_i)$ to $ker(d'_i)$, which is exactly what we need to get an induced map.

5) a) Prove that a short exact sequence $0 \to C' \to C \to C'' \to 0$ of complexes gives a long exact sequence in homology

$$\cdots \to H_i(C') \to H_i(C) \to H_i(C'') \to H_{i-1}(C') \to \cdots$$

The maps $H_i(C') \to H_i(C) \to H_i(C'')$ are induced by $C' \to C \to C''$ as in 4), and the "connecting map" $H_i(C'') \to H_{i-1}(C')$ is TBD.

Proof: I'm going to introduce a framework for working with complexes which is extremely useful before doing this problem. The idea as it pertains to this particular problem is to first work with a class of sequences, not short exact, for which the connecting homomorphism will be obvious, and then deal with short exact sequences by relating them to these. The techniques and language are modeled on algebraic topology.

Given a complex A and $n \in \mathbf{Z}$, define a shifted complex $A[n]$ by $(A[n])_i = A_{n-i}$ and $d^{A[n]} = (-1)^n d^A$; given a map $f : A \to B$, define the map $f[n] : A[n] \to B[n]$ by $f[n]_i = f_{n-i}$. Given two complexes A and B, define a tensor product complex $A \otimes B$ by $(A \otimes B)_n = \oplus_{i+j=n} A_i \otimes B_j$ with differential defined by the "Leibniz rule":

$$d(a_i \otimes b_j) = d(a_i) \otimes b_j + (-1)^i a_i \otimes d(b_j).$$

Note that, with this definition, $A[n] = R[n] \otimes A$, where $R[n]$ is the complex with R in the n^{th} slot and zero elsewhere (so the differential is necessarily always zero).

Now, we want to set up a notion of homotopy between maps of chain complexes modeled on the case of topological spaces. For this let I denote the chain complex of the unit interval simplicial set, i.e. I_0 is the free R-module on two elements, p_0 and p_1; I_1 is the free R-module on one element e; every other I_n is zero; and the differential is defined by $d(e) = p_1 - p_0$. We have two maps of chain complexes $R \to I$ where R is given by R in degree 0 and zero elsewhere (in general, if M is an R-module, we'll just use M to denote the complex with M in degree 0 and 0 elsewhere; for these things a map between chain complexes is the same as a map between the original modules, which shows that this is reasonable). The two maps are $p_i : R \to I$ defined by $r \mapsto rp_i$.

For any complex A, these give two maps $A = R \otimes A \xrightarrow{p_i \otimes A} I \otimes A$, which we'll also denote by p_i. Now, given two chain complexes A and B and two maps $f, g : A \to B$, we say that f and g are *homotopic*, and write $f \sim g$, if there is an $h : I \otimes A \to B$ with $h \circ p_0 = g$ and $h \circ p_1 = f$. If you unravel this, it comes down to saying that there are maps of R-modules $h_n : A_{n-1} \to B_n$ for each $n \in \mathbf{Z}$ such that $f - g = dh + hd$.

If you didn't follow the above discussion of tensor products, you can just remember this definition and be OK.

Now, using either of the two definitions, you can easily check things like:

- Homotopy is an equivalence relation;
- $f \sim g$ if and only if $f - g \sim 0$;
- The set of $f \sim 0$ is an R-submodule of $Hom(A, B)$;
- For any $f : A \to B$ and $g : B \to C$, if $f \sim 0$ or $g \sim 0$, then $f \circ g \sim 0$.

This lets us define an R-module $[A, B] = Hom(A, B)/\sim$, and R-bilinear composition laws $[A, B] \times [B, C] \to [A, C]$. We can talk about diagrams commuting up to homotopy, and define a homotopy equivalence to be a map $f : A \to B$ such that there is a $g : B \to A$ with $f \circ g \sim id_B$ and $g \circ f \sim id_A$; equivalently, for every K, the map $[K, A] \to [K, B]$ induced by composing with f is a bijection, or for every K the map $[B, K] \to [A, K]$ is a bijection.

Let's note a special case of the notion of homotopy. What is $[R, A]$? Well, giving a map of complexes from R to A is the same as giving an element a_0 of A_0, but not just any element: because maps of complexes commute with the differential, it has to satisfy $da_0 = 0$, i.e. it must be a cycle. You can work out that two such things are homotopic as maps if and only if they are coboundaries, so in fact $[R, A] = H_0(A)$. And if you believe that, you also shouldn't have trouble believing that $[R[n], A] = H_n(A)$ for all $n \in \mathbf{Z}$ (or again, $[R, A[-n]] = H_n(A)$). So the homology groups are special cases of homotopy classes of maps into complexes. Note also that, under this correspondence, the composition law $[R[n], A] \to [R[n], B]$ for a given $f : A \to B$ is just the induced map on H_i as in problem 4'. In particular, we deduce that *homotopic maps induce the same map on homology*, an important thing to remember. It implies that homotopy equivalences induce isomorphisms on homology.

Another definition by analogy with topological spaces: given a map $f : A \to B$, define a complex $C(f)$, called the *mapping cone* of f, as the colimit of the following diagram:

$$\begin{array}{ccccc} & & A & \xrightarrow{f} & B \\ & & \downarrow{\scriptstyle p_1} & & \\ A & \xrightarrow{p_0} & I \otimes A & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Thus, giving a map $C(f) \to K$ is the same as giving a map $B \to K$ together with a homotopy between its composition with f and the zero map. If you work it out explicitly, you'll see that $(C(f))_n = B_n \oplus A_{n-1}$ with differential $d(b, a) = (db + f(a), -da)$. Again, if you don't know anything about colimits in the category of complexes (and why should you?), you can just remember this definition and be perfectly fine. In that case, though, make sure to check that this actually does define a complex, i.e. $d^2 = 0$.

This notion of cone is the one that we'll be exploiting. Define a *triangle* to be a system of maps

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{e} A[1]$$

Of course, such a thing can be extended in both directions just by shifting:

$$\ldots \longrightarrow B[-1] \xrightarrow{g[-1]} C[-1] \xrightarrow{e[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{e} A[1] \xrightarrow{f[1]} B[1] \longrightarrow \ldots$$

In particular, for any complex K, it gives a long sequence (what once was f now denotes pre-composition with f, and so on):

$$\ldots \longrightarrow [K, C[-1]] \xrightarrow{e[-1]} [K, A] \xrightarrow{f} [K, B] \xrightarrow{g} [K, C] \xrightarrow{e} [K, A[1]] \xrightarrow{f[1]} \ldots$$

Even more particularly ($K = R$), it gives a long sequence of homology groups

$$\ldots \longrightarrow H_1(C) \xrightarrow{e} H_0(A) \xrightarrow{f} H_0(B) \xrightarrow{g} H_0(C) \xrightarrow{e} H_{-1}(A) \xrightarrow{f} \ldots$$

Now, using mapping cones, let's introduce a class of triangles for which these long sequences are guaranteed to be exact.

Any map $f : A \to B$ gives a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C(f) \xrightarrow{e} A[-1],$$

where $g(b) = (b, 0)$ and $e(b, a) = a$. Call such a triangle "pre-distinguished". I claim that for any pre-distinguished triangle, the associated long sequence on $[K, -]$ is exact. In particular, the long sequence of homology groups is exact. First we check exactness at $[K, B]$ by hand, then we use a trick.

Exactness at $[K, B]$ amounts to saying that for $\phi : K \to B$, the condition $g \circ \phi \sim 0$ is equivalent to the condition that there exist a map $\psi : K \to A$ with $\phi \sim f \circ \psi$. But saying $g \circ \phi \sim 0$ is saying there's a map $h : K_{n-1} \to C(f)_n = B_n \oplus A_{n-1}$ satisfying certain conditions; similarly saying $\phi \sim f \circ \psi$ for some $\psi : K \to A$ is saying there are $K_{n-1} \to A_{n-1}$ and $K_{n-1} \to B_n$ satisfying certain conditions; it's clear how to make the data correspond, and as for the conditions, if you write them out explicitly you'll see they then coincide. In brief, this is trivial.

Now, the trick is this. Exactness at $[K, B]$ clearly gives exactness at $[K, B[n]]$ for all $n \in \mathbf{Z}$; we could either argue by replacing K by $K[-n]$ or by noting that the triangle

$$A[n] \xrightarrow{f[n]} B[n] \xrightarrow{g[n]} C(f)[n] = C(f[n]) \xrightarrow{h[n]} A[n+1]$$

is also pre-distinguished. So what we need to check is exactness at $[K, A]$ and $[K, C(f)]$; if we get those, by a similar shifting argument we'll be done. But we can also do those by a shifting argument! The idea is this: though the rotated triangles

$$B \xrightarrow{g} C(f) \xrightarrow{e} A[1] \xrightarrow{-f[1]} B[1]$$

and

$$C(f)[-1] \xrightarrow{e[-1]} A \xrightarrow{-f} B \xrightarrow{g} C(f)$$

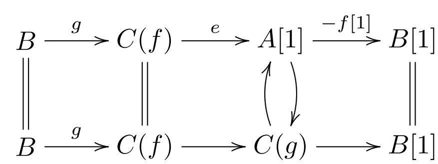
are not pre-distinguished triangles, they are *close enough* to the pre-distinghuished triangles (respectively)

$$B \xrightarrow{g} C(f) \to C(g) \to B[1]$$

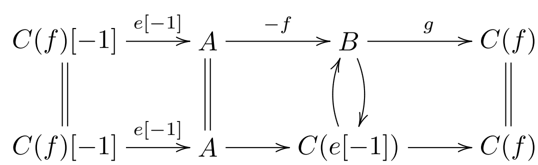
and

$$C(f)[-1] \xrightarrow{e[-1]} A \to C(e[-1]) \to C(f),$$

in the following sense: we have diagrams of chain complexes



where the map $A[1] \to C(g)$ is $a \mapsto (0, a, -f(a))$ and the map $C(g) \to A[1]$ is $(b, a, b') \mapsto a$ (you should check that these commute with the differential), and



where the map $B \to C(e[-1])$ is $b \mapsto (0, b, 0)$ and the map $C(e[-1]) \to B$ is $(a, b, a') \mapsto b - f(a)$ (again, check that these commute with the differential). The property we care about for these diagrams is that they are commutative *up to homotopy*, and that the curved vertical maps are mutually inverse homotopy equivalences.

Assuming this, applying $[K, -]$ we get a diagram which is literally commutative and for which the vertical arrows are all literally isomorphisms, so the first diagram then reduces exactness at $C(f)$ in the long sequence for the original triangle to exactness at $C(f)$ in the predistinguished triangle for g, which we checked above, and similarly for the second diagram and exactness at A. To check the claim, note that it suffices to simply see that the curved arrows are mutually inverse homotopy equivalences, since it's trivial to check that the up arrows make the middle squares commute and the down arrows make the right arrows commute, let alone up to homotopy; thus everything will commute (up to homotopy) provided they're inverse (up to homotopy). Now, in both cases, going down then up is already the identity, so we just need to check that going up then down is homotopic to the identity. For the first diagram, define $h(b, a, b') = (0, 0, b)$ (all you really can do if you want to map degree n to degree $n + 1$), and for the second diagram define $h(a, b, a') = (0, 0, a)$ (ditto). It's easy to check that these give homotopies between the identity map and the map given by going up and then down, which checks the claim, and finishes the proof that a pre-distinguished triangle gives a long exact sequence in $[K, -]$ for any complex, and in particular in homology. Here's a corollary:

Corollary: A map $f : A \to B$ between chain complexes induces an isomorphism on $[K, -]$ if and only if $[K, C(f)[n]] = 0$ for all n (or $[K[n], C(f)] = 0$ for all n). In particular, f induces an isomorphism on homology if and only if $C(f)$ has trivial homology.

Proof: Immediate from the long exact sequence.

Now, let's finally tackle short exact sequences of complexes. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be one. Then we have a commutative diagram

$$\begin{CD} A @>f>> B @>>> C(f) \\ @| @| @VV\varphi V \\ A @>f>> B @>g>> C \end{CD}$$

where $\varphi : (b, a) \mapsto g(b)$ (to check this commutes with the differential, just need that g does and that $g \circ f = 0$). I claim that this map φ is an isomorphism on homology. Assuming this, the fact that the top line has a connecting map and a long exact sequence is homology (seen above) implies that the bottom one does as well. To check the claim, we use the corollary: it suffices to see that $C(\varphi)$ has trivial homology.

We easily compute the differential on $C(\varphi)$ as $(c, b, a) \mapsto (g(b) + d(c), f(a) - d(b), d(a))$; what we need, then is that if $g(b) + d(c) = 0$, $f(a) - d(b) = 0$, and $d(a) = 0$ then there are (c', b', a') such that $g(b') + d(c') = c$, $f(a') - d(b') = b$, and $d(a') = a$. Take $c' = 0$, b' arbitrary in the fiber $g^{-1}(\{c\})$, and a' the unique element of $f^{-1}(\{b + d(b')\})$—which exists since $g(b + d(b')) = g(b) + g(d(b')) = g(b) + d(g(b')) = d(b) + d(c) = 0$. Then the first two equations are obvious from choice, and to check the last we can check after applying f, since this is injective, and indeed

$$f(d(a')) = d(f(a')) = d(b + d(b')) = d(b) + d(d(b')) = d(b) = f(a).$$

This (finally) finished the proof. □

Remark: It's also true that if $A \to B \to C(f) \to A[1]$ is a pre-distinguished triangle, then for any complex K there is a long exact sequence for $[-, K]$ applied to the triangle as well (by the same argument, it suffices to check exactness at A; you can do this without trouble). I could use this to streamline the proof of the lemma in the following problem, following the technique of the lemma of problem 8, but... why don't I leave the fun to you. You might also enjoy showing that if $0 \to A \xrightarrow{f} B \to C \to 0$ is a short exact sequence which is degree-wise split, then the map $C(f) \to C$ defined above is not just a homology isomorphism, but a homotopy equivalence.

6) Let C (resp. C') be a projective resolution of M (resp. M'). Prove that a map $\alpha : M \to M'$ induces a map of complexes $\tilde{\alpha} : C \to C'$.

Proof: Let's reinterpret and generalize the question. Recall that, for a module M, we also denote by M the complex which is M in degree zero and 0 elsewhere. Then a projective resolution of M is just a complex P consisting of projective modules living in non-negative degrees together with a map of complexes $P \to M$ which is a *quasi-isomorphism*, i.e. an isomorphism on homology. What we want is that a map $M \to M'$ induces a map $P \to C'$ making the obvious diagram commute.

Lemma: Let P be a complex consisting of projective modules and A and B two complexes which are bounded above, i.e. $A_n = B_n = 0$ for n small enough. If

$f : A \to B$ is a quasi-isomorphism, then every map $P \to B$ has an up-to-homotopy lift to $P \to A$; furthermore this lift is unique up to homotopy.

Proof: The desired conclusion is that $[P, A] \to [P, B]$ is a bijection, so by the corollary of the previous problem what we need is that $[P, C(f)] = 0$. By the same lemma, we know that $C(f)$ has trivial homology. And $C(f)$ is certainly bounded above if A and B are, so we've reduced to showing that if A is bounded above and has trivial homology, then $[P, A] = 0$, i.e. every map $f : P \to A$ is homotopic to zero.

So we need maps $h_n : P_{n-1} \to A_n$ such that $f = hd + dh$. If $A_n = 0$ we just take $h_n = 0$, as we must; now we proceed inductively. Suppose we have h_n, and we want h_{n+1}, which needs to satisfy $dh_{n+1} = f_n - h_n d$. If we can just show that $f_n - h_n d$ lands in $im(d)$, this will follow from P_n being projective: it's just a lifting question. But since A has trivial homology, it will suffice to check that $d(f_n - h_n d) = 0$. And indeed,

$$d(f_n - h_n d) = df_n - dh_n d = df_n - (f_{n-1} - h_{n-1} d)d = df_n - f_{n-1}d - h_{n-1}d^2 = 0 - 0 = 0.$$

This finishes the proof. □

Now let's put the lemma to work. Let $P \to M$ be a projective resolution of M, and $C' \to M'$ an arbitary resolution of M'. Then given a map $M \to M'$, we can apply the lemma to $P \to M \to M'$, and deduce an up-to-homotopy lift $P \to C'$. But if two maps $P \to M'$ are homotopic, then they are equal, since P lives in nonnegative degrees and M' lives in degree zero, so any h_n must be the zero map. Thus it is an actual lift, as desired.

7) Prove that $Tor_i(M, N)$ does not depend on the projective resolution of M.

Proof: Let $P \to M$ and $P' \to M$ be two projective resolutions of M. By the lemma of the previous problem, we get maps $P \to P'$ and $P' \to P$, unique up to homotopy, making the obvious diagram commute up to homotopy (or not, by the argument in the previous problem). But then by the uniqueness up to homotopy of the lifting in the lemma, both compositions $P \to P' \to P$ and $P' \to P \to P'$ must be homotopic to the identity, which means that $P \to P'$ is a homotopy equivalence. But then the same is true of the induced map $P \otimes N \to P' \otimes N$, since we can just define homotopies here by tensoring the original ones with N (the homotopy relation is just an identity involving the abelian group operation on morphisms, and tensoring maps preserves this structure). Homotopy equivalences give homology isomorphisms (recall from problem 5), so this implies, on taking homology, that the two ways of computing Tor are isomorphic.

8) Prove that $Tor_i(M, N)$ can also be computed by taking a projective resolution of N, applying $M \otimes -$, and taking homology.

Proof: Kudos to some of you for sloughing out the double complex argument; I hope you never have to do something like that again (either because you'll know about spectral sequences or you'll know the argument I'm about to present). In fact we'll show that Tor can be computed by a *flat* resolution in either variable. By problem 2, this is more general. Let $E \to M$ be a flat resolution of M, and $F \to N$ a flat resolution of N. We want to show that $E \otimes N$ has isomorphic homology to $F \otimes M$. Here is the lemma, analogous to that of two problems ago.

Lemma: Let E be a bounded-above complex of flat modules and $f : A \to B$ a quasi-isomorphism between bounded-above complexes. Then $E \otimes A \to E \otimes B$ is also a quasi-isomorphism (see the beginning of problem 5 for the definition of tensor product of complexes).

Proof: First note that tensoring with E commutes with mapping cones, i.e. $E \otimes C(f) \simeq C(E \otimes f)$. This has nothing to do with E being flat or bounded above, and is easy to just check (if you remember the colimit description, you can do it essentially without work; it turns out that $E \otimes -$ has a right adjoint. Can you find it? It's the thing you'd want to consider if you want to do this problem with Ext instead of Tor). Anyway, by the corollary in problem 5, this reduces us to proving that if A (is bounded above and) has trivial homology, then so does $E \otimes A$. This is extremely plausible: flat things are supposed to preserve exactness under tensor. But since E is a whole complex, $E \otimes A$ doesn't look all that simple. So let's be a little more clever before digging in.

Denote by $\tau_{\leq n} E$ the complex which is E_i in degree $i \leq n$ and 0 elsewhere, with E's differentials wherever the differentials aren't forced to be zero. Then we have the map $E_n[n-1] \to \tau_{\leq n-1} E$ given in degree n by the differential $d : E_n \to E_{n-1}$; it's easy to check that the mapping cone of this is isomorphic to $\tau_{\leq n} E$, so we have a pre-distinguished triangle (or something isomorphic to one, which is just as good):

$$E_n[n-1] \to \tau_{\leq n-1} E \to \tau_{\leq n} E \to E_n[n]$$

But just as above, since tensoring commutes with mapping cones, this is also (isomorphic to) predistinguished:

$$E_n[n-1] \otimes A \to (\tau_{\leq n-1} E) \otimes A \to (\tau_{\leq n} E) \otimes A \to E_n[n] \otimes A.$$

Then since A is exact and E_{n+1} is flat, $E_{n+1}[n] \otimes A$ exact (i.e. has trivial homology), so by the long exact sequence the map $(\tau_{\leq n-1} E) \otimes A \to (\tau_{\leq n} E)$ is a quasi-isomorphism. Thus $(\tau_{\leq n-1} E) \otimes A$ has the same homology as $(\tau_{\leq n} E) \otimes A$, and by induction all the $(\tau_{\leq n} E) \otimes A$ have the same homology. But since E is bounded above, some $\tau_{\leq n} E$ is zero; thus all the $(\tau_{\leq n} E) \otimes A$ have trivial homology. On the other hand, since A is bounded above, say $A_n = 0$ for $n < N$, right from the definition we have

$$\tau_{\leq n}(E \otimes A) = \tau_{\leq n}((\tau_{\leq n-N} E) \otimes A),$$

which implies that $H_{n-1}(E \otimes A) = H_{n-1}((\tau_{\leq n-N} E) \otimes A) = 0$; this being true for all n, we have the desired result. □

Now, the lemma makes the problem trivial. First, a small observation: perhaps despite initial appearances, for complexes A and B, we have $A \otimes B \simeq B \otimes A$; it's easy to see that $a_i \otimes b_j \mapsto (-1)^{ij} b_j \otimes a_i$ furnishes an isomorphism of complexes. Thus we can apply the lemma with tensoring on either side. Now, since $F \to N$ is a quasi-isomorphism, the lemma says that $E \otimes F$ and $E \otimes N$ have isomorphic homology; but then since $E \to M$ is a quasi-isomorphism, so do $E \otimes F$ and $M \otimes F$. Thus $E \otimes N$ and $M \otimes F$ have isomorphic homology, as desired.

Note two corollaries: first, $Tor_i(M, N) \simeq Tor_i(N, M)$ for all N, M, and second, $Tor_i(M, N) = 0$ for $i > 0$ if either M or N is flat (a flat module has itself as a flat resolution).

9) Let $x \in A$ be a non-zerodivisor. Prove that

$$Tor_1(A/(x), M) = \{m \in M \mid xm = 0\}.$$

Proof: First we claim that a short exact sequence of modules $0 \to M \to N \to P \to 0$ gives a long exact sequence in Tor groups

$$\ldots \to Tor_i(M, L) \to Tor_i(N, L) \to Tor_i(P, L) \to Tor_{i-1}(M, L) \to \ldots$$

for all modules L. Indeed, we can just take a flat resolution F of L; then

$$0 \to M \otimes F \to N \otimes F \to P \otimes F \to 0$$

is a short exact sequence of complexes (simply because each term of F is flat), and its long exact sequence in homology yields the desired. (We could also do a different argument by taking a resolution of each of M, N, and P... but why make life difficult?)

Given this, consider the short exact sequence $0 \to A \xrightarrow{\cdot x} A \to A/(x) \to 0$, where $\cdot x$ is injective precisely because x is a non-zerodivisor. A part of its long exact sequence reads

$$Tor_1(A, M) \to Tor_1(A/(x), M) \to A \otimes M \xrightarrow{\cdot x} A \otimes M.$$

But, uh oh, I probably should check that the map $A \otimes M \to A \otimes M$ given as the Tor_0 term in the long exact sequence is actually the same as the $\cdot x$ map. I would've been more careful about all this, except you guys don't know categories. Bummer. OK, here goes.

A more general claim is that if $f : M \to N$ is a map of modules, then the induced map on $Tor_0(-, L)$ via the above long exact sequence is identified with the map $f \otimes L : M \otimes L \to N \otimes L$. Part of this claim should be that, in fact, there is a natural way to identify $Tor_0(M, L)$ with $M \otimes L$ for any modules M and L.

This might be more naturally seen using the "different argument" I referred to above, but I've made my choice, and I'm sticking to it. Take a flat resolution F of L; part of this is saying that L is the cokernel of $F_1 \to F_0$, and since tensoring is right exact, $M \otimes L$ is the cokernel of $M \otimes F_1 \to M \otimes F_0$, but this is just the same as $H_0(M \otimes F) = Tor_0(M, L)$. That gives the identification of $M \otimes L$ with $Tor_0(M, L)$. To see that, through this identification, the map $Tor_0(M, L) \to Tor_0(N, L)$ coincides with the map $M \otimes L \to N \otimes L$, it suffices to stare at the commutative digram

$$\begin{array}{ccccc} M \otimes F_1 & \longrightarrow & M \otimes F_0 & \longrightarrow & M \otimes L \\ \downarrow & & \downarrow & & \downarrow \\ N \otimes F_1 & \longrightarrow & N \otimes F_0 & \longrightarrow & N \otimes L \end{array}$$

where all the down maps are induced by f.

OK, to recap, we do indeed have that part of the long exact sequence written above, with the $\cdot x$ labeling justified. But $Tor_1(A, M) = 0$ since A is flat, so that sequence just says that $Tor_1(A/(x), M)$ is the kernel of $\cdot x$ on $A \otimes M$, or, what is the same, of $\cdot x$ on M, as desired.

10) (AM 2.24) Let M be an A-module. Show that the following are equivalent:

(1) M is flat;
(2) $Tor_n(M, N) = 0$ for all N;
(3) $Tor_1(M, N) = 0$ for all N.

Proof: We remarked after the end of problem 9 that (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), suppose M is such that $Tor_1(M, N) = 0$ for all N. Take a short exact sequence $0 \to P \to Q \to N \to 0$; then part of the long exact sequence is

$$0 = Tor_1(M, N) \to P \otimes M \to Q \otimes M \to N \otimes M \to 0,$$

where we checked that the maps are as expected in problem 9; this shows, by definition, that M is flat.

11) (AM 2.26) Let N be an A-module. Show that N is flat if and only if $Tor_1(A/I, N) = 0$ whenever I is a finitely generated ideal in A.

Proof: "Only if" by the previous problem. For "if", the key fact will be that $Tor_i(-, N)$ commutes with directed limits. Sigh. I guess I should prove this. It comes from the same being true of $- \otimes N$. In fact, that guy commutes with arbitrary co-limits (who is responsible for a directed *limit* being an example of a *co-limit*??), since it has the right adjoint of $Hom(N, -)$. Now, take a directed limit $M = \lim M_\alpha$, and choose a flat resolution F of N. Then by the result for tensors, we have $M \otimes F = \lim(M_\alpha \otimes F)$ in the obvious (degree-wise) sense (still directed); we want to see that the same formula holds when we take homology. What we need to show is that if a complex C is the filtered limit of complexes C_α, then, firstly, every element of $H_n(C)$ comes from some element of $H_n(C_\alpha)$ for some α, and secondly, that if something in $H_n(C_\alpha)$ gives zero in $H_n(C)$, then it gives zero in $H_n(C_\beta)$ for some β dominating α. For the first, if we have something in $H_n(C)$, it's represented by an $c \in C_n$ with $dc = 0$; this c comes from some $c_\alpha \in (C_\alpha)_n$, but $d(c_\alpha)$ might not be zero. However, it goes to zero in C_{n-1}, so it must be zero on some level β dominating α, and on that level we have a cycle which goes to c. So there you go. For the second thing, if a cycle $c_\alpha \in (C_\alpha)_n$ gets sent to a boundary in C_n, say is hit by c', then c' also comes from some $(C_\gamma)_{n+1}$; then dc' and c go to the same thing in the end, so on some level containing α and γ they must become equal, as desired.

OK, that argument was probably not up to your standards of mathematical exposition. I'm sorry. If you don't like it you can do it yourself, geez. Actually, I don't buy that anyone is really reading this, so I'm not going to sweat it. Anyway, let's get back to the show.

Suppose $Tor_1(A/I, N) = 0$ for all finitely generated ideals I. We'll successively show that $Tor_1(A/J, N) = 0$ for *all* ideals J, that $Tor_1(M, N) = 0$ for all finitely generated modules M, and finally that $Tor_1(M, N) = 0$ for all modules M, so that N is flat by the previous problem. For the first step, let J be an arbitary ideal; I claim $Tor_1(A/J, N) = 0$. Consider

$$0 \to J \to A \to A/J \to 0.$$

The long exact sequence in $Tor(-, N)$ gives

$$0 \to Tor_1(A/J, N) \to J \otimes N \to A \otimes N,$$

since A is free, hence flat (or projective), implying $Tor_1(A, N) = 0$. So we want $J \otimes N \to A \otimes N$ to be injective. But J is the directed limit of its finitely generated sub-ideals J_α, and for those, by the long exact sequence for $0 \to J_\alpha \to A \to A/J_\alpha \to 0$, the hypothesis $Tor_1(A/J_\alpha, N) = 0$ gives that $J_\alpha \otimes N \to A \otimes N$ is injective; since $- \otimes N$ commutes with directed limits, the claim becomes trivial.

For the next step, let's show that $Tor_1(M, N) = 0$ for all finitely generated modules M. We induct on the number of generators. If there's just one, M is isomorphic to A/I for some I (the kernel of the induced map $A \to M$), so that case is handled. If there's more than one, say there are n, write M' for the submodule generated some $n-1$ of them. Then the quotient M/M' is just generated by one element, the one we threw out. The long exact sequence for $0 \to M' \to M \to M/M' \to 0$ gives

$$Tor_1(M', N) \to Tor_1(M, N) \to Tor_1(M/M', N),$$

and the outer guys are zero by inductive hypothesis, so the middle one is too, as desired. Finally, an arbitrary module is the directed limit of its finitely generated submodules, so the last step follows from the above cantankerously-proved statement about $Tor_i(-, N)$ commuting with directed limits.

12) Let $(A, \mathfrak{m})$ be a Noetherian local ring and M finitely generated over A. Show that M is flat if and only if M is free.

Proof: Certainly free things are flat, since direct sums of exact sequences are exact. Now suppose M flat. Just as in problem 3, we can use Nakayama's lemma to get a surjective map $A^n \to M$ which is an isomorphism on tensoring with $k = A/\mathfrak{m}$. Let K be the kernel; then we have a short exact

$$0 \to K \to A^n \to M \to 0,$$

on which the long exact sequence gives

$$Tor_1(M, k) \to K \otimes k \to A^n \otimes k \to M \otimes k \to 0.$$

But $Tor_1(M, k) = 0$ since M is flat, and the map $A^n \otimes k \to M \otimes k$ is an isomorphism, as we guaranteed through choice; thus $K \otimes k = 0$; since A is noetherian and

$K \subseteq A^n$, we know that K is finitely generated, so by Nakayama we deduce $K = 0$, so that our map $A^n \to M$ is an isomorphism, i.e. M is free.

13) Conclude from the previous problems that if A is noetherian and M finitely generated over A then M is flat if and only if M is projective.

Proof: From problem 2, projective modules are always flat. So assume M flat. Then for each prime $P \in Spec(A)$, the A_P-module M_P is flat (if N is an A_P-module, then it's also an A-module, and $N \otimes_{A_P} M_P \simeq N \otimes_A M$ naturally). And it is certainly finitely generated, since M is. So the previous problem says that M_P is locally free; then problem 4 says that M is projective.

14) Assume no finiteness hypothesis. Find a ring A and a flat module M which is not projective.

Example: Let $A = \mathbf{Z}$, and $M = \mathbf{Z}_{(2)}$. Localizations are flat, so M is flat. On the other hand, it is not projective: consider the surjection $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$. We have the obvious map $\mathbf{Z}_{(2)} \to \mathbf{Z}/2\mathbf{Z}$, but it doesn't have a lift: indeed, $1 \in \mathbf{Z}_{(2)}$ would have to go to some odd (hence nonzero) integer a in $\mathbf{Z}$; then if p is an odd prime not dividing a, poor $1/p \in \mathbf{Z}_{(2)}$ can't go anywhere.