Problem Set 11

Math 221 Fall 2007

In this problem set we will study the module of relative differentials. You may skip any one problem of your choice.

Derivations and differentials.
Let $A \to B$ be a ring homomorphism making B an A-algebra, and let M be a B-module.

An A-*derivation* of B into M is a map $d : B \to M$ such that

a) $d(b + b') = db + db'$,
b) $d(bb') = bdb' + b'db$, and
c) $da = 0$ for $a \in A$.

The set of A-derivations of B into M is a B-module denoted $\mathrm{Der}_A(B, M)$.

Let I denote the kernel of the map $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$. Define a map $d : B \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$. We will denote I/I^2 by $\Omega_{B/A}$.

(1) Prove that $d : B \to \Omega_{B/A}$ has the following universal property: for any B-module M and any A-derivation $D : B \to M$ there is a unique B-module homomorphism $f : \Omega_{B/A} \to M$ so that $D = f \circ d$.

The module $\Omega_{B/A}$ is called the *module of differentials*, or *module of Kähler differentials* of B over A. It is characterized (up to unique isomorphism) by the universal property. One can also define $\Omega_{B/A}$ by taking the free B-module generated by $\{db \mid b \in B\}$ and quotienting out by the relations a), b), c) above. By Problem (1), there is a canonical isomorphism $\mathrm{Hom}_B(\Omega_{B/A}, M) \simeq \mathrm{Der}_A(B, M)$ of B-modules.

(2) Suppose $B = A[x_1, x_2, \ldots, x_n]$. Verify that $\Omega_{B/A}$ is the free B-module of rank n generated by $dx_1, \ldots, dx_n$.

(3)a) If A' is another A-algebra and $B' = B \otimes_A A'$ then

$$\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B' \simeq \Omega_{B/A} \otimes_A A'.$$

b) If $S \subset B$ is a multiplicative set, then $\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$.

The first fundamental exact sequence Let $A \to B \to C$ be rings and ring homomorphisms. Then there is a natural exact sequence of C-modules

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

The second fundamental exact sequence Let B be an A-algebra, $I \subset B$ an ideal and $C = B/I$. Then there is a natural exact sequence of C-modules

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

where the first map sends $b \in I/I^2$ to $db \otimes 1$.

(4)(a) Explicitly write down the maps in both fundamental exact sequences, and verify (quickly) that they are indeed C-module homomorphisms.
(b) Pick one of the two sequences and prove the exactness.

(5) Calculate $\Omega_{B/A}$ when $A = k$ is a field and $B = k[x, y]/(y^2 - x^3)$.

(6) Deduce from (2) and the exact sequences that if B is either a finitely-generated A-algebra, or the localization of a finitely-generated A-algebra, then $\Omega_{B/A}$ is a finite B-module.

Differentials and regular local rings

(7) Let (B, m) be a local ring containing a field k which maps isomorphically onto B/m under $B \to B/m$. Show that the first map $m/m^2 \to \Omega_{B/k} \otimes_B k$ in the second fundamental exact sequence is an isomorphism.

Thus if $\Omega_{B/k}$ is free of rank $\dim(B)$ then B is a regular local ring. The converse is also true if one makes some assumptions concerning k (but needs more work).

The modules $\Omega_{B/A}$ are algebraic versions of differential forms. In algebraic geometry, one defines the *Zariski tangent space* at the point corresponding to the local ring (B, m) to be the k-vector space $\mathrm{Hom}_k(m/m^2, k)$. This agrees with the intuition

$$\text{regular} \iff (\#\ \text{tangent directions} = \text{dimension}) \iff \text{smooth}$$

If you know some differential geometry, you may want to try to interpret the fundamental exact sequences geometrically. For example, I/I^2 in the second sequence should be thought of as a counterpart of the *conormal space*.