Problem Set 6

Math 221 Fall 2007

This problem set will cover some long overdue homological algebra. Let A be a ring. Modules will be modules over A unless otherwise specified.

Remark: In many places "Noetherian ring and finitely generated module" can be replaced by the weaker "finitely presented module".

Projective modules.
A module M is *projective* if for any surjection $\alpha : N \twoheadrightarrow L$ of modules and map $g : M \to L$ one can find a lift $\tilde{g} : M \to N$ such that $g = \alpha \circ \tilde{g}$.

1) Prove that the following are equivalent.

a) M is projective.
b) M is a direct summand of a free module.
c) For any surjection $N \twoheadrightarrow L$, the natural map $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, L)$ is surjective.
d) Every surjection $\alpha : N \twoheadrightarrow M$ splits, that is, there is a map $\beta : M \to N$ such that $\alpha \circ \beta$ is the identity.

2) Prove that a projective module is flat.

3) Suppose $A = (A, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated projective A-module. Prove that M is free. Hint: use Nakayama's Lemma to find $m_1, \ldots, m_n \in M$ such that their images in $M/\mathfrak{m}M$ is a basis. Define a map $A^n \to M$ by $e_i \mapsto m_i$, and use Nakayama's Lemma again to show that this is an isomorphism. (The result is also true without any finiteness hypotheses, but is quite a bit harder – no Nakayama's Lemma!)

4) Suppose A is Noetherian and M a finitely generated A-module. Then M is projective if and only if M_P is a free A_P-module for each prime $P \subset A$.

Complexes. A *complex* is a sequence

$$C = \cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} C_{i-2} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

of maps and modules such that the composition of any two (adjacent) maps is zero. The homology modules $H_i(C)$ are defined by $H_i(C) = \ker(d_i)/\mathrm{im}(d_{i+1})$. Thus $H_i(C) = 0$ if C is exact at C_i. A map $\alpha : C \to C'$ between two complexes is a map $\alpha_i : C_i \to C'_i$ for each i such that all the obvious squares commute.

4) Prove that $\alpha : C \to C'$ induces a map $\alpha_* : H_i(C) \to H_i(C')$ on homology for each i.

If $0 \to C' \to C \to C'' \to 0$ is a sequence of complexes, we say that it is *exact* if each $0 \to C'_i \to C_i \to C''_i \to 0$ is.

5)a) Prove that an exact sequence of complexes induces a long exact sequence in homology:

$$\cdots H_1(C) \to H_1(C'') \to H_0(C') \to H_0(C) \to H_0(C'') \to 0.$$

You only have to verify this for the part of the sequence shown. The key is to figure out the *connecting homomorphism* $H_1(C'') \to H_0(C')$ (looking at b) might help).
b) Explain the Remark after [AM, Proposition 2.10].

Let M be a module. A *resolution* $C \to M$ of M is a complex C together with a map $C_0 \to M$ such that $\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$ is exact. The resolution is free (or projective) if each C_i is. It is clear that free resolutions and hence projective resolutions exist.

6) Let C (resp. C') be a projective resolution of M (resp. M'). Prove that a map $\alpha : M \to M'$ induces a map of complexes $\tilde{\alpha} : C \to C'$.

Tor. Let M, N be modules. Pick a projective resolution $C \to M$. The sequence

$$C \otimes N = \cdots \to C_{i+1} \otimes N \to C_i \otimes N \to C_{i-1} \otimes N \to \cdots$$

is a complex. We define $\mathrm{Tor}_i(M, N) = H_i(C \otimes N)$. Thus $\mathrm{Tor}_0(M, N) = M \otimes N$.

7) Prove that $\mathrm{Tor}_i(M, N)$ does not depend on the projective resolution $C \to M$ of M.

8) Prove that $\mathrm{Tor}_i(M, N)$ can also be computed by taking a projective resolution of N, applying $M \otimes -$, and taking homology. Hint: let $C \to M$ and $D \to N$ by projective resolutions. Write down the "double complex" $C \otimes D$ and chase homology groups diagonally.

9) Let $x \in A$ be a non-zerodivisor. Prove that

$$\mathrm{Tor}_1(A/(x), M) = \{m \mid xm = 0\}.$$

10) Do Exercise 2.24 in [AM].

11) Do Exercise 2.26 in [AM].

12) Let $(A, \mathfrak{m})$ be a Noetherian local ring and M finitely generated over A. Show that M is flat if and only if M is free.

13) Conclude from the previous problems and results in class the following: if A is Noetherian and M finitely generated over A then M is flat if and only if M is projective.

14) Assume no finiteness hypotheses. Find a ring A and a flat module M which is not projective.