Fall 2023 Math 538 Problem Set 1

Due Wed Sep 13, at the beginning of class.

You are allowed to make the blanket assumption that $F = \mathbb{C}$ or $F = \mathbb{R}$.

1. Let I be an ideal of L. Prove that each member of the derived series $D^n(I)$ or lower central series $C^n(I)$ is an ideal of L.

2. In class, we defined Lie algebras that preserve a bilinear form. Explicitly describe the symplectic Lie algebra for the skew-symmetric bilinear form θ, taking values on a basis $e_1, \ldots, e_{2n}$ of $\mathbb{C}^{2n}$ as follows

$$\theta(e_i, e_j) = \begin{cases} \delta_{i,2n+1-j} & \text{if } i < j \\ -\delta_{i,2n+1-j} & \text{if } j \leq i. \end{cases}$$

Explicitly describe the orthogonal Lie algebra in the case of the following symmetric bilinear form ω, taking values on a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ as follows

$$\omega(e_i, e_j) = \delta_{ij}.$$

Explicitly describe the orthogonal Lie algebra in the case of the following symmetric bilinear form η, taking values on a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ as follows

$$\eta(e_i, e_j) = \delta_{i,n+1-j}.$$

(It may help to split into the even and the odd case.)

3. (H1.12) Let L be a Lie algebra and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\operatorname{ad} x$ is a subalgebra.

4. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis of $\mathfrak{sl}(2, F)$. Compute the matrices of $\operatorname{ad} e, \operatorname{ad} h, \operatorname{ad} f$ relative to this basis.

5. (H2.6) Prove that $\mathfrak{sl}(3, F)$ is simple. (Assume $F = \mathbb{C}$.)

6. (H3.2) Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

7. Suppose that $L = [L, L]$ and $\dim L = 3$. Show that L is simple.

8. Suppose that $x \in \mathfrak{gl}_n$ has distinct eigenvalues $a_1, a_2, \ldots, a_n$. Find the n^2 (counted with multiplicity) eigenvalues of $\operatorname{ad} x$.