Fall 2023 Math 538 Problem Set 2

Due Wednesday Oct 4, at the beginning of class.

You are allowed to make the blanket assumption that $F = \mathbb{C}$, unless the problem explicitly states otherwise.

1. (H4.1) Use Lie's theorem to prove that $L = \mathfrak{sl}_n$ is semisimple, as follows. Let R be the radical of L. Show that there is a change of basis such that $R \subset \mathfrak{b}_n$. However, R must be closed under transposing matrices (why?). Use this to conclude that R must be 0.

2. Let $x, y \in \mathrm{End}(V)$. Give examples to show that the equalities $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y_n$ can fail.

3. Show that if L is nilpotent, then the Killing form of L is identically zero. Find an example of a non-nilpotent Lie algebra with identically zero Killing form.

4. (H5.4) Let L be the three-dimensional solvable Lie algebra with basis $\{x, y, z\}$, and Lie bracket $[x, y] = z, [x, z] = y, [y, z] = 0$. Compute the radical of the Killing form of L.

5. (H5.5) Let $L = \mathfrak{sl}_2$. Compute the basis of L dual to the standard basis $\{e, h, f\}$ relative to the Killing form.

6. (H6.1) Using the previous problem, write down the Casimir element of the adjoint representation of $L = \mathfrak{sl}_2$. Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}_3$, first computing dual bases relative to the trace form.

7. (H6.3) Show that if L is solvable, then every irreducible representation of L is one-dimensional.

8. (H6.6) Assume L is simple. Show that any two symmetric, non-degenerate, invariant bilinear forms on L are proportional. [Use Schur's Lemma.]

9. (H6.7) Use the previous problem to show that the Killing form κ on $L = \mathfrak{sl}_n$ is given by $\kappa(x, y) = 2n\mathrm{Tr}(xy)$.