Fall 2023 Math 538 Problem Set 3

Due Wednesday Oct 18, at the beginning of class.

You are allowed to assume that $F = \mathbb{C}$, unless the problem explicitly states otherwise. Lie algebras and their representations are finite-dimensional, unless stated otherwise.

1. (Semi-direct products)

(a) Let L and M be Lie algebras and assume that we have a Lie algebra morphism $\eta : M \to \mathrm{Der}(L)$. Show that the vector space $L \oplus M$ equipped with the bracket

$$[(l, m), (l', m')] = ([l, l']_L + \eta(m)(l') - \eta(m')(l), [m, m']_M)$$

is a Lie algebra and $L \oplus 0$ is an ideal.

(b) Conversely, let X be a Lie algebra, $L \subset X$ an ideal, $M \subset X$ a subalgebra, such that $X = L \oplus M$ as a vector space. Show that $m \mapsto \mathrm{ad}\,(m)|_L$ is a Lie algebra morphism $\eta : M \to \mathrm{Der}(L)$ and that the bracket on X coincides with the bracket on $L \oplus M$ defined above.

(c) Show that M is an ideal if and only if η is trivial.

2. (Characters) Define the character of an $\mathfrak{sl}_2$-module V to be the Laurent polynomial

$$\chi_V(t) = \sum_i (\dim V_i) t^i,$$

where V_i denotes the i-weight space of V.

(a) What are the characters of the irreducible $\mathfrak{sl}_2$-modules?

(b) Let V, W be two $\mathfrak{sl}_2$-modules. Show that $\chi_V(t) = \chi_W(t)$ if and only if $V \cong W$.

(c) Let V, W be two $\mathfrak{sl}_2$-modules. Show that $\chi_{V\otimes W}(t) = \chi_V(t)\chi_W(t)$.

(d) Use characters to deduce a tensor product rule for $\mathfrak{sl}_2$. That is, give the decomposition into irreducible representations of the tensor product of two irreducible $\mathfrak{sl}_2$-modules.

3. (Plethysm)

(a) Deduce from the previous exercise that the odd and even parts of every character $\chi_V = \sum_i a_i t^i$ are symmetric and unimodal; i.e., $a_i = a_{-i}$, $a_0 \geq a_2 \geq a_4 \geq \cdots$, and $a_1 \geq a_3 \geq a_5 \geq \cdots$.

(b) Let std denote the standard two-dimensional representation of $\mathfrak{sl}_2$. Show that if $W = \mathrm{Sym}^m(\mathrm{Sym}^n(\mathrm{std}))$, then the coefficient of t^{2k-mn} in $\chi_W(t)$ is the number of partitions of k into at most m parts, each of size $\leq n$ (i.e., integer solutions of $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and $\sum \lambda_i = k$). Thus (a) implies that these numbers are symmetric and unimodal as a function of k.

(c) Show that $\mathrm{Sym}^m(\mathrm{Sym}^n(\mathrm{std})) \cong \mathrm{Sym}^n(\mathrm{Sym}^m(\mathrm{std}))$ as $\mathfrak{sl}_2$-modules. [For an arbitrary finite-dimensional $\mathfrak{sl}_k$-module V, Foulkes conjecture (still open!) states that $\mathrm{Sym}^m(\mathrm{Sym}^n(V))$ contains $\mathrm{Sym}^n(\mathrm{Sym}^m(V))$ as a $\mathfrak{sl}_k$-submodule when $m > n$.]

(d) Show that $t^{mn}\chi_W(t)$ is the Gaussian polynomial $(t^2)_{m+n}/((t^2)_m(t^2)_n)$, where $(q)_k := (1-q)(1-q^2)\cdots(1-q^k)$. (Hint: use (b) and induction.) This shows that Gaussian polynomials are symmetric and unimodal, an old theorem of Sylvester.

4. (H8.5) Prove that if L is semisimple and $\mathfrak{h}$ is a maximal toral subalgebra, then $\mathfrak{h}$ is its own normalizer in L.

5. (H8.6) Let $L = \mathfrak{sl}_n$. Compute the basis of L dual to the standard basis relative to the Killing form. The standard basis consists of the matrix elements $\{e_{ij} \mid 1 \leq i \neq j \leq n\}$ and the diagonal matrices $\{e_{ii} - e_{i+1,i+1} \mid 1 \leq i \leq n-1\}$.

6. (H8.7) Assume that L is semisimple and $\mathfrak{h} \subset L$ a maximal toral subalgebra.

(a) Prove that $C_L(h)$ is reductive for all $h \in \mathfrak{h}$.

(b) Prove that it is possible to choose h so that $C_L(h) = \mathfrak{h}$.

(c) Characterize when the situation in (b) happens for $\mathfrak{sl}_n$.