Fall 2023 Math 538 Problem Set 4

Due Wednesday Nov 8, at the beginning of class.

In the following, Φ is a root system in a Euclidean space E, with W as Weyl group and Δ denotes a base of Φ.

1. (H9.4) Prove that the Weyl groups of A_2, B_2, G_2 are the dihedral groups of order 6, 8, 12.

2. (H9.6) Prove that W is a normal subgroup of AutΦ. (See Section 9.2 for the definition of AutΦ.)

3. (see H9.7) Let $E' \subset E$ be such that $\Phi' := E' \cap \Phi$ is non-empty. Show that Φ' is a root system in span(Φ').

4. (H9.9) Let (E, Φ) be a possibly non-reduced root system, that is, it satisfies (R1), (R3), (R4) but not necessarily (R2). Prove that the only possible multiples of $\alpha \in \Phi$ that can be in Φ are $\pm\alpha/2, \pm\alpha, \pm 2\alpha$. Verify that $\{\alpha \in \Phi \mid 2\alpha \notin \Phi\}$ is a root system.

5. (H10.6) Prove that the map $w \mapsto (-1)^{\ell(w)}$ is a group homomorphism $W \to \{\pm 1\}$.

6. (see H10.9)

(a) Prove that there is a unique element $w_0 \in W$ of maximum length. Moreover, $w_0^2 = 1$, $w_0\Phi^+ = \Phi^-$, and $\ell(w_0) = |\Phi^+|$.

(b) Show that $-w_0$ is an automorphism of Φ that permutes Δ, and compute this automorphism for A_2, B_2, G_2.

7. (H10.11) Prove that Φ is irreducible if and only if $\Phi^\vee$ is.

8. (H11.2) Calculate the determinants of the Cartan matrices (using induction on ℓ for types A_ℓ-D_ℓ), which are as follows:

$$A_\ell : \ell + 1; B_\ell : 2; C_\ell : 2; D_\ell : 4; E_6 : 3; E_7 : 2; E_8, F_4, G_2 : 1.$$