Fall 2023 Math 538 Problem Set 5

Due Fri Nov 17, at the beginning of class.

1. Let Φ be a root system. Recall that for $\alpha, \beta \in \Phi$ if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi \cup \{0\}$ and if $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Phi \cup \{0\}$. Prove the converse of these implications assuming that Φ is simply-laced (all roots have the same length). Show by example that this converse fails if R is not simply-laced.

2. (H12.3) Let $\Phi \subset E$ satisfy (R1),(R3),(R4), but not (R2). Suppose that Φ is irreducible. Prove that Φ is the union of root systems of type B_n, C_n in E (if $\dim E = n > 1$), where the long roots of B_n are also the short roots of C_n. (This is called the non-reduced root system of type BC_n in the literature.)

3. (H13.5) If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is W-invariant. Therefore, we obtain a homomorphism $\phi : \mathrm{Aut}\,\Phi/W \to \mathrm{Aut}(\Lambda/\Lambda_r)$. Prove that ϕ is injective, then deduce that $-1 \in W$ if and only if $\Lambda_r \supset 2\Lambda$. Show that $-1 \in W$ for precisely the irreducible root systems $A_1, B_\ell, C_\ell, D_\ell (\ell \text{ even}), E_7, E_8, F_4, G_2$.

4. (H14.2) Let $L = \mathfrak{sl}_2$. If $\mathfrak{h}, \mathfrak{h}'$ are any two maximal toral subalgebras of L, prove that there exists and automorphism of L mapping $\mathfrak{h}$ onto $\mathfrak{h}'$.

5. (H17.1) Prove that if $\dim L < \infty$ then $U(L)$ has no zero divisors. [Hint: Use the fact that the associate graded algebra is isomorphic to a polynomial algebra.]

6. (H17.3) If $x \in L$, extend $\mathrm{ad}\,x$ to an endomorphism of $U(L)$ by defining $\mathrm{ad}\,x(y) = xy - yx$ for $y \in U(L)$. If $\dim L < \infty$, prove that each element of $U(L)$ lies in a finite dimensional L-submodule. [If $x, x_1, \ldots, x_m \in L$, verify that $\mathrm{ad}\,x(x_1 \cdots x_m) = \sum_{i=1}^m x_1 x_2 \cdots \mathrm{ad}\,x(x_i) \cdots x_m$.]

7. (H20.2c) Let $L = \mathfrak{sl}_2$, with standard basis (x, y, h). Show that $1 - x$ is not invertible in $U(L)$, hence lies in a maximal left ideal I of $U(L)$. Set $V = U(L)/I$, so V is an irreducible L-module. Prove that the images of $1, h, h^2, \ldots$ are all linearly independent in V (so $\dim V = \infty$), using the fact that

$$(x-1)^r h^s \equiv \begin{cases} 0 (\mathrm{mod}\ I), & r > 2 \\ (-2)^r r! \cdot 1 (\mathrm{mod}\ I), & r = s. \end{cases}$$