Fall 2023 Math 538 Problem Set 6

Due Wed Dec 6, at the beginning of class.

1. Show that the universal Casimir element c_L is central in $U(L)$. (Hint: look at the argument we previously used for the Casimir elements c_ϕ of a representation ϕ.)

2. Verify the identity $\sum_i \lambda(h_i)\lambda(k_i) = (\lambda, \lambda)$ where $\lambda \in \mathfrak{h}^*$ and $\{h_i\}, \{k_i\}$ are dual bases of $\mathfrak{h}$.

3. (H20.3) Describe weights and highest weight vectors for the natural (or defining) representations of the linear Lie algebras of types A_ℓ, B_ℓ, C_ℓ described in Section 1.2. [You may use the statements in Section 19.2, for example, that the classical algebras are simple and the diagonal matrices in them form a maximal toral subalgebra. Root vectors are given in Section 1.2. You will need to make a choice of positive roots.]

4. (H21.6) Let $V = V(\lambda)$ with $\lambda \in \Lambda^+$. Prove that V^* is isomorphic as a L-module to $V(-w_0\lambda)$, where $w_0 \in W$ is the longest element.

5. (H21.7) Recall that $\Pi(V)$ denotes the set of weights of V. Let $V = V(\lambda)$, $W = V(\mu)$ with $\lambda, \mu \in \Lambda^+$. Prove that $\Pi(V \otimes W) = \{\nu + \nu' \mid \nu \in \Pi(V), \nu' \in \Pi(\mu)\}$, and that

$$\dim(V \otimes W)_\pi = \sum_{\pi = \nu + \nu'} \dim V_\nu \cdot \dim W_{\nu'}.$$

Deduce that $V(\lambda + \mu)$ occurs exactly once as a direct summand of $V \otimes W$.

6. (H21.11) Let $V = F^n$ be the defining representation of $\mathfrak{sl}_n$. Show that the exterior powers $\Lambda^k V$ for $1 \leq k \leq n-1$ are irreducible representations of $\mathfrak{sl}_n$, and find their highest weight.