Winter 2014 Math 566 Problem Set 1
Due Friday January 17

Unless explicitly specified, all enumerative problems (asking for a proof of the equality of two quantitites) can be done either combinatorially (e.g. a direct bijection), or by algebraic methods (e.g. generating functions). BONUS points are awarded for providing solutions of **both** types.

1. Prove both algebraically and combinatorially the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

for $n \geq 1$.

2. A composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of n is an ordered sequence of positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$.

   (a) Prove that the number of compositions of n is equal to 2^{n-1}.

   (b) Prove that the number of compositions of n with an even number of even parts is equal to the number of compositions of n with an odd number of even parts.

   (c) Prove that the total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

   (d) Fix positive integers k, n. Show that

$$\sum \alpha_1 \cdots \alpha_k = \binom{n+k-1}{2k-1}$$

   where the sum is over all compositions $(\alpha_1, \ldots, \alpha_k)$ of n into exactly k parts.

3. Prove that $\sum_{k=1}^{n} k\,A(n,k) = \frac{1}{2}(n+1)!$.

4. A derangement $w \in S_n$ is a permutation such that $w_i \neq i$ for each i (that is, w has no fixed points). Let $D(n)$ be the number of derangements in S_n. Prove using generating functions that

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right).$$

(Optional: also prove this using a direct bijection.)