Winter 2014 Math 566 Problem Set 2
Due Friday January 25

1. Let $s_i \in S_n$ for $i \in \{1, 2, \ldots, n-1\}$ be the simple transposition which swaps i and $i+1$ and leaves all other numbers fixed. Recall that S_n is a group with respect to composition of bijections. Thus $s_i w$ is obtained from w by swapping the values i and $i+1$ in the one-line notation for w. For example, $s_1 s_2 = 231 \in S_3$. Also if $w = 72148635$ then $s_5 w = 72148536$.

   (a) Prove that S_n is generated as a group by $s_1, s_2, \ldots, s_{n-1}$.

   (b) Prove that the elements s_i satisfy the relations

   $$\begin{aligned} s_i^2 &= 1 & & \\ s_i s_j &= s_j s_i & & \text{if } |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & & \text{for } i = 1, 2, \ldots, n-2. \end{aligned}$$

   (c) (Bonus:) Prove that these are all the relations. That is, S_n is isomorphic to the abstract group with generators $s_1, s_2, \ldots, s_{n-1}$ and these relations.

   (d) Define the length $\ell(w) \in \mathbb{Z}_{\geq 0}$ of $w \in S_n$ to be the smallest integer ℓ such that there exists an expression

   $$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

   for w as a product of ℓ simple transpositions. (The length of the identity permutation is defined to be 0.) For example, $321 = s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3.
   Show that $\ell(w) = \mathrm{inv}(w)$.

2. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a composition of n such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. For example, the partitions of 5 are

   $$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1).$$

   Let $p(n)$ denote the number of partitions of n.

   (a) Argue that we have

   $$\sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

   (b) Prove that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

3. A permutation $w \in S_n$ is even if $(-1)^{\mathrm{inv}(w)} = 1$, and is odd otherwise. Show that the total number of cycles of all even permutations in S_n and the total number of cycles of all odd permutations of S_n differ by $(-1)^n (n-2)!$.

4. In class it was mentioned that the Eulerian number $A(n,k)$ has the following interpretation:

$$A(n,k) = \text{Vol}((x_1, x_2, \ldots, x_n) \in [0,1]^n \cap \{k-1 \le x_1 + x_2 + \cdots + x_n \le k\}) \tag{1}$$

as the normalized volume of a slice of the hypercube in n dimensions. (Normalized means that the slice for $k = 1$, which is a simplex, is defined to have volume 1.)

We shall prove the above statement in this problem.

(a) For each i, define $y_i = x_1 + x_2 + \cdots x_i$, and define $z_i = y_i - \lfloor y_i \rfloor \in [0,1)$. Check that for nearly all points $(x_1, x_2, \ldots, x_n) \in [0,1]^n$, the vector $(z_1, z_2, \ldots, z_n)$ determines $(x_1, x_2, \ldots, x_n)$. Conversely, nearly all $(z_1, z_2, \ldots, z_n) \in [0,1)^n$ comes from a unique point $(x_1, x_2, \ldots, x_n) \in [0,1]^n$. Here "nearly all" means except for a union of lower-dimensional pieces.

(b) Say that $z = (z_1, z_2, \ldots, z_n)$ has descent set D if $z_i > z_{i+1}$ exactly when $i \in D$.

Prove that if $(x_1, x_2, \ldots, x_n)$ satisfies $k - 1 \le x_1 + x_2 + \cdots + x_n \le k$ if and only if the descent set of $(z_1, z_2, \ldots, z_n)$ has size $k - 1$.

(c) Prove that the map $\phi : (x_1, x_2, \ldots, x_n) \mapsto (z_1, z_2, \ldots, z_n)$ is a measure-preserving map from $[0,1]^n$ to $[0,1)^n$. (Hint: the domain can be split up into pieces where the map ϕ is linear. To check that a linear map is measure-preserving one just needs to show that the determinant is ± 1. Finally, note that we can always ignore some lower-dimensional pieces since they have measure zero.)

(d) Conclude the equality (1), by cutting up the z-space appropriately.