Winter 2014 Math 566 Problem Set 3
Due Friday February 7

1. Let $D = \frac{\partial}{\partial x}$ be the differential operator in x. Prove that

$$(e^x D)^n = e^{nx} \sum_k s(n,k) D^k.$$

and

$$x^n D^n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) (xD)^k.$$

(Recall that for us $s(n,k)$ denotes the *signless* Stirling number.)

2. A poset is *connected* if its Hasse-diagram is connected. Equivalently, a connected poset is not non-trivally a disjoint union of two posets.

Let P be a finite poset with a $\hat{0}$ and a $\hat{1}$. Suppose that every open interval

$$(x,y) := \{z \in P \mid x < z < y\}$$

is either an anti-chain, or connected. Prove that P is graded.

3. In the previous problem set we defined partitions. Let $P(n)$ be the set of partitions of n. Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ are in $P(n)$. Then we declare that $\lambda \leq \mu$ in *dominance order* if

$$\lambda_1 \leq \mu_1, \qquad \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \qquad \lambda_1 + \lambda_2 + \lambda_3 \leq \mu_1 + \mu_2 + \mu_3, \cdots$$

For example $(4,1) > (3,2)$. (In the above inequalities, we should think of a partition as an integer sequence with many trailing zeroes.)

   (a) Prove that $(P(n), \leq)$ is a poset.
   (b) The dual $(Q, \leq)$ of a poset $(P, \leq)$ is the poset with the same underlying set $P = Q$, such that if $x \leq y$ in P then $y \leq x$ in Q. Prove that $(P(n), \leq)$ is isomorphic to its dual.
   (c) Prove that $(P(n), \leq)$ is a lattice.

4. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a collection of (affine) hyperplanes in $\mathbb{R}^d$. Here "affine" means that the hyperplanes don't have to pass through the origin.

We say that $\mathcal{A}$ is a *generic* arrangement if

$$\dim(H_{i_1} \cap \cdots \cap H_{i_k}) = d - k$$

for $k \in [d+1]$ and $\{i_1, \ldots, i_k\} \subset [n]$. The emptyset is taken to have dimension -1.

   (a) Let $d = 2$ or $d = 3$. Find a formula for the number of components of the complement $\mathbb{R}^d \setminus \mathcal{A}$.

(b) (Bonus) Find the formula for general d.

5. The *inversion poset* P_w of a permutation $w \in S_n$ has underlying set $[n]$, and partial order $i \prec j$ if and only if $i < j$ and $w_i < w_j$. A permutation w is 3142-avoiding if there do not exist $i < j < k < \ell$ such that $w_j < w_\ell < w_i < w_k$. Similarly we define 2413-avoiding.

   Let Z denote the four-element zig-zag poset, consisting of elements $\{x, y, z, u\}$ such that $x < y$, $y > z$, and $z < u$. Prove that P_w contains no induced subposets isomorphic to Z if and only if w is 3142-avoiding and 2413-avoiding.