Winter 2014 Math 566 Problem Set 4
Due Friday February 21

1. (a) Suppose L and L' are lattices. Show that $L \times L'$ is also a lattice.
   (b) Prove that for any positive integer n, the lattice D_n is isomorphic, as a lattice, to a product of chains. For which n is the lattice D_n isomorphic to the boolean lattice B_m?
   (c) Which of the properties in $\{graded, modular, atomic, distributive\}$ are preserved under taking direct products of lattices?

2. Let L be a finite lattice. Prove that L is semimodular if and only if

$$s \wedge t \lessdot t \implies s \lessdot s \vee t$$

   for all $s, t \in L$.

3. Let L be a finite lattice.
   (a) Prove that

$$s \vee (t \wedge u) \leq (s \vee t) \wedge u$$

   for all $s, t, u \in L$ satisfying $s \leq u$.
   (b) Prove that L is modular if and only if equality holds:

$$s \vee (t \wedge u) = (s \vee t) \wedge u$$

   for all $s, t, u \in L$ with $s \leq u$. (Hint: for the "only if" direction, compare the ranks $\rho(s \vee (t \wedge u))$ and $\rho((s \vee t) \wedge u)$. For the "if" direction, we can prove semimodularity as follows. Suppose $(x \wedge y) \lessdot y$. Apply the above equation to deduce that an element z in the interval $[x, x \vee y]$ must be equal to either x or $x \vee y$.)
   (c) Prove that a distributive lattice is modular.

4. A *linear extension* of a finite poset P with n elements is an order-preserving bijection from P to an n-element chain. (A chain with n elements has *length* $n-1$.) Let $e(P)$ denote the number of linear extensions of P.

   For each $p \in P$, define $h_p = \#\{q \mid q \leq p\}$. Prove that

$$e(P) \geq \frac{n!}{\prod_{p \in P} h_p}.$$

   (Bonus: when does equality hold?)

5. Recall that Π_n denotes the partition lattice.
   (a) Prove that Π_n is graded, and compute the number of maximal chains in Π_n.

(b) Two maximal chains C and C' in Π_n are *equivalent* if they are obtained from each other by renaming the elements $1, 2, \ldots, n$; that is, they are related by an element of S_n acting on Π_n. For example $(1|2|3|4) \lessdot (12|3|4) \lessdot (12|34) \lessdot (1234)$ is equivalent to $(1|2|3|4) \lessdot (24|1|3) \lessdot (24|13) \lessdot (1234)$, and are related by the permutation $1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 3$.

Prove that the number of equivalence classes of maximal chains in Π_n is equal to the Euler number E_{n-1}. (Use the relation to flip-equivalence classes of increasing binary trees.)