Winter 2014 Math 566 Problem Set 5
Due Friday March 14

1. Let P be a finite poset, with a $\hat{0}$ and $\hat{1}$. Prove that $\sum_{s\leq t}\mu(s,t)=1$.

2. Let L be a finite lattice. Let $a(n)$ denote the number of ordered n-tuples $(x_1,x_2,\ldots,x_n)\in L^n$ satisfying $x_1\wedge x_2\wedge\cdots\wedge x_n=\hat{0}$. Prove, using a calculation in the Mobius algebra, that

$$a(n)=\sum_{x\in L}\mu(\hat{0},x)(b(x))^n$$

where $b(x)=\#\{y\in L\mid x\leq y\}$ denotes the number of elements in L bigger than or equal to x.

3. Let P be a finite graded poset with a $\hat{0}$ and $\hat{1}$. We say that P is *Eulerian* if each interval $[s,t]$ where $s<t$ has the same number of elements with odd rank as elements with even rank.

   (a) What do intervals of length 2 (that is, $[s,t]$ where $\rho(t)=\rho(s)+2$) in Eulerian posets look like?

   (b) Verify that the Boolen algebra B_n is Eulerian.

   (c) Prove that a poset is Eulerian if and only if the Mobius function is given by $\mu(s,t)=(-1)^{\rho(t)-\rho(s)}$.

   (d) Suppose P and Q are Eulerian. Show that $P\times Q$ is also Eulerian.

   (e) Suppose P and Q are finite graded posets with $\hat{0}$ and $\hat{1}$. Prove that the poset $(P-\hat{1})\oplus(Q-\hat{0})$ is also Eulerian. Recall that the operation $\oplus$ puts everything in $(Q-\hat{0})$ above everything in $(P-\hat{1})$.

4. Let $\mathcal{A}$ be an arrangement in the n-dimensional vector space V whose normals span a subspace X, and let $\mathcal{B}$ be another arrangement in V whose normals span a subspace Y. Suppose that $X\cap Y=\{0\}$. Prove that

$$\chi_{\mathcal{A}\cup\mathcal{B}}(t)=t^{-n}\chi_{\mathcal{A}}(t)\chi_{\mathcal{B}}(t).$$

5. Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^n$ consisting of the n hyperplanes

$$x_1=x_2,x_2=x_3,\ldots,x_{n-1}=x_n,x_n=x_1.$$

Compute the characteristic polynomial $\chi_{\mathcal{A}}(t)$ and the number $r(\mathcal{A})$ of regions of $\mathcal{A}$.