Winter 2014 Math 566 Problem Set 6
Due Friday March 28

1. A collection $\mathcal{M}$ of k-element subsets of $[n]$, satisfies the *exchange axiom* if: given $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.

   (a) Suppose $\mathcal{M}$ satisfies the exchange axiom. Show that $\mathcal{M}$ satisfies the *dual exchange axiom*: if $I, J \in \mathcal{M}$ and $j \in J$ there exists $i \in I$ such that $(I - \{i\} \cup \jmath\}) \in \mathcal{M}$.

   (b) Suppose $\mathcal{M}$ satisfies the exchange axiom. Show that $\mathcal{M}$ satisfies the *symmetric exchange axiom*: if $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that both $(I - \{i\} \cup \{j\})$ and $(J - \{j\} \cup \{i\})$ belong to $\mathcal{M}$.

   (c) Suppose M is a matroid on $[n]$, with independent sets $\mathcal{I}(M)$. For an integer $k \geq 1$, let $\mathcal{M}_k = \{I \in \mathcal{I}(M) \mid |I| = k\}$. Prove that $\mathcal{M}_k$ is either empty, or it satisfies the exchange axiom.

2. Show that if F and F' are two flats of a matroid M, then so is $F \cap F'$.

3. In class I argued that the intersection poset $L(\mathcal{A})$ of the braid arrangement $\mathcal{A}$ was isomorphic to the partition lattice Π_n. Describe in a similar manner the intersection poset $L(\mathcal{A}_G)$ for any graph G on vertex set $[n]$.

4. Suppose L is a geometric lattice. We call an element $x \in L$ *modular* if for all $y \in L$ we have an equality

$$\rho(x) + \rho(y) = \rho(x \wedge y) + \rho(x \vee y).$$

   Prove that all atoms are modular.

5. Suppose that L is a geometric lattice and $[x, y] \subset L$ is an interval. Show that $[x, y]$ is also a geometric lattice.

6. Consider the hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ consisting of the hyperplanes $x_i - x_j = 0$ for $1 \leq i < j \leq n$, the hyperplanes $x_i + x_j = 0$ for $1 \leq i < j \leq n$, and the hyperplanes $x_i = 0$ for $1 \leq i \leq n$. Prove that the characteristic polynomial of $\mathcal{A}$ is given by

$$\chi_{\mathcal{A}}(t) = (t-1)(t-3)(t-5)\cdots(t-2n+1).$$