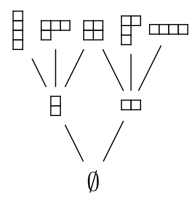
Winter 2014 Math 566 Problem Set 7
Due Monday April 21

1. Define a poset P by starting from the empty partition and instead of adding boxes, we allow dominoes (both vertical and horizontal) to be added. The first few levels look like:



   (a) Prove that P is a 2-differential poset.

   (b) What is the analogue of $\sum_{|\lambda|=n}(f^\lambda)^2 = n!$ for P? State your answer in terms of tableaux, not chains.

   (c) Is P a lattice? A distributive lattice?

   (d) What is the relationship between P and $\mathbb{Y}^2$?

2. A 2-core is a partition λ such that no domino (vertical or horizontal) can be removed from it, still leaving a partition. In other words, λ is a 2-core if there does not exist a partition $\mu \subset \lambda$ such that λ is obtained from μ by adding a domino.

   For example, $\emptyset$ and (1) are 2-cores.

   (a) What are all the 2-cores?

   (b) Let λ be any partition. Start removing dominoes from λ, while always remaining a partition. Eventually we will arrive at a 2-core μ. Prove that μ does not depend on how we removed dominoes.

   For example, suppose $\lambda = (2, 2)$ is a two-by-two square. We can remove two vertical dominoes, or two horizontal dominoes, but we will always end up at $\emptyset$.

3. Define the Fibonacci poset F as follows. The elements of F are words using the letters 1 and 2, including the empty word (which is the minimal element of F). A word u is covered by v if

   (a) v is obtained from u by inserting a "1" somewhere prior to the left most "1" in u (or if u has no 1's, we can insert a "1" anywhere), or

   (b) we change the leftmost "1" in u into a "2" to get v.

   Prove that F is a differential poset. Is F a lattice?

4. Suppose λ is a partition of n. Prove that

$$\sum_{\mu \gtrdot \lambda} f^{\mu} = (n+1) f^{\lambda}.$$

5. (Optional bonus question) Suppose T is a SYT. The reading word reading(T) is obtained by reading the entries of T from left to right along each row, starting with the first row. Obviously reading(T) is a permutation. The sign of T is the sign (that is, (-1) to the power of the number of inversions) of the reading word of T. Prove that

$$\sum_{|T|=n} \text{sign}(T) = 2^{\lfloor n/2 \rfloor}.$$

Hint: try to put signs into Young's lattice, and study the relation $DU + UD = I$.