Problem Set 3 Math 637 Winter 2012

You may use any theorems proved in class, or proved in the textbook [Kir] before the corresponding exercise.

(1) (a) Let $H \subset G$ be a closed Lie subgroup. Prove that G/H is Hausdorff as a topological space.
    (b) Find a smooth action of $\mathbb{R}$ on $\mathbb{R}^2$, so that all the orbits are embedded curves, but $\mathbb{R}^2/\mathbb{R}$ is not Hausdorff.

(2) Let G be a non-compact Lie group with an element $g \in G$ such that the set $\{g^k \mid k \in \mathbb{Z}\} \subset G$ is dense in G. Show that $G \simeq \mathbb{Z}$. (Optional: what happens if G is compact?)

(3) [Kir 3.6]

(4) [Kir, 3.14]

(5) [Kir, 4.1]

(6) [Kir, 4.2]

(7) Let $\mathfrak{g} = sl(2, \mathbb{C})$ and $\mathfrak{g}_0 \subset \mathfrak{g}$ a real Lie subalgebra such that $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$. Necessarily, $\mathfrak{g}_0$ is a three-dimensional real subspace. Classify the possible $\mathfrak{g}_0$'s up to isomorphism.

    You may want to proceed as follows:
    (a) Prove that there is a non-zero element $X \in \mathfrak{g}_0$ such that the 3×3 matrix $\mathrm{ad}(X)$ is diagonalizable. (If you can't prove this, you may assume it and keep going.)
    (b) Check that the eigenvalues of $\mathrm{ad}(X)$ sum to 0.
    (c) Show that one of the eigenvalues of $\mathrm{ad}(X)$ is 0, so that the other two are either both real, or conjugate imaginary numbers.
    (d) In the case that all eigenvalues are real, show that $\mathfrak{g}_0$ must be isomorphic to $sl(2, \mathbb{R})$.
    (e) Figure out what happens when we have conjugate imaginary eigenvalues.