Problem Set 5 Math 637 Winter 2012

(1) Prove that every compact Lie group G is isomorphic to a closed subgroup of $U(n)$ for some n.

(2) Let V be a representation of $sl(2, \mathbb{C})$, with weight spaces $V(k) = \{v \in V \mid h \cdot v = kv\}$.
   (a) Suppose $k \geq 1$. Show that the action of f gives an injective map from $V(k) \to V(k-2)$.
   (b) Suppose $k \geq 1$. Show that the action of f^k gives a bijective map from $V(k)$ to $V(-k)$.

   (This kind of behavior goes by the name of "Hard Lefschetz Theorem" in geometry.)

(3) Let X be a Hausdorff space with a continuous action of a compact Lie group G. Show that the quotient space X/G is Hausdorff.

(4) (a) Give an example of a maximal abelian subgroup H of a compact connected group G which is not a maximal torus.
   (b) Show that the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subset G$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$. Is every maximal abelian Lie subalgebra of $\mathfrak{g}$ the Lie algebra of some maximal torus?

(5) The *representation ring* $R(G)$ of a compact group G is the subspace of the ring of complex-valued class functions, spanned over $\mathbb{Z}$ by the characters of complex finite-dimensional representations of G.

   Let $G = U(n)$. Define the class functions $e_k : U(n) \to \mathbb{C}$ taking $g \in U(n)$ to the k-th elementary symmetric function in the eigenvalues of g. Thus $e_1(g) = \mathrm{tr}(g)$ and $e_n(g) = \det(g)$. Prove that

$$R(G) = \mathbb{Z}[e_1, e_2, \ldots, e_n, e_n^{-1}].$$

(6) Give an example of a connected compact group G and $g \in G$ such that the centralizer $C_G(g)$ is not connected. Show that the identity component $C_G(g)_0$ is the union of the maximal tori containing g.

(7) Show that for a continuous class function $f \in C^0(SU(2), \mathbb{C})$, one has

$$\int_{SU(2)} f(g)dg = \frac{2}{\pi} \int_0^{\pi} f(t(\theta)) \sin^2(\theta)\, d\theta$$

where

$$t(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Establish this formula in two different ways: (a) using the Weyl integration formula, and (b) expanding f into irreducible characters, and using the orthogonality relations of characters.