Problem Set 6 Math 637 Winter 2012

(1) [Kir, 7.1] (The definition of a non-reduced root system is at the beginning of Chapter 7.)
(2) [Kir, 7.2(1)] Show also that the root systems $R \subset V$ and $R^\vee \subset V^*$ have isomorphic Weyl groups.
(3) [Kir, 7.3]
(4) Let $\alpha \in R$ be a root. Show that there are vectors $e_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$, $f_\alpha \in \mathfrak{g}_{\mathbb{C},-\alpha}$, and $h_\alpha \in \mathfrak{h}_{\mathbb{C}}$ so that the map

$$e_\alpha \mapsto e, f_\alpha \mapsto f, h_\alpha \mapsto h$$

gives an isomorphism of $\text{span}\{e_\alpha, f_\alpha, h_\alpha\}$ with $sl(2,\mathbb{C})$.
(5) Let $f : H \to G$ be a surjective homomorphism of compact connected Lie groups.
   (a) Show that if $T \subset H$ is a maximal torus, then $f(T) \subset G$ is a maximal torus.
   (b) Suppose that $\ker(f) \subset Z(H)$, where $Z(H)$ denotes the center of H. Show that f induces an isomorphism of Weyl groups of H and G.
(6) Let G be a compact connected Lie group with $\dim(G) = 4$ and $\text{rank}(G) = 2$. Show that we have

$$1 \to Z(G) \to G \to SO(3) \to 1$$

where $Z(G)$ is isomorphic to either S^1 or $S^1 \times \mathbb{Z}/2\mathbb{Z}$.
(7) Prove the classification of rank 2 root systems given in class. Explicitly describe all the Weyl groups.