Problem Set 7 Math 637 Winter 2012

(1) In class we fudged over the calculation of the invariant inner product for the Lie algebra of the torus. Let's remedy this.
   (a) [Kir, 7.15]
   (b) Suppose (R, V) is an irreducible root system. Prove that there is a unique W-invariant inner product on V up to scalar factor.
   (c) Suppose (R, V) is a root system. Prove that there is precisely one W-invariant inner product on V, satisfying the identity
$$(v_1, v_2) = \sum_{\alpha \in R} (\alpha, v_1)(\alpha, v_2).$$
(2) The *Killing form* of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form defined by
$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}X\ \mathrm{ad}Y).$$
   (a) Explain why κ is invariant under the adjoint representation of G.
   (b) Calculate κ for $\mathfrak{g} = sl(2, \mathbb{C})$.
   (c) Suppose $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G. Show that
$$\kappa(X, X) = -\sum_{\alpha \in R} \alpha(X)^2$$
   for $X \in \mathfrak{h} = Lie(T)$.
   (d) Suppose G is a semisimple compact connected group. Prove that κ is negative definite when restricted to $\mathfrak{h}$. Use $-\kappa$ to identify $\mathfrak{h}$ with $\mathfrak{h}^*$, and show that $-\kappa$ induces the inner product of Problem 1c) for $(R, \mathfrak{h}^*)$.
(3) [Kir, 7.4]
(4) [Kir, 7.5] This group is the center of the corresponding simply-connected semisimple compact group.
(5) [Kir, 7.9]