**Problem Set 1**
**Due on Friday Sept 13**

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

As usual, TNN means totally nonnegative, and TP means totally positive. Also $[n] := \{1, 2, \ldots, n\}$.

**Problem 1.**

(1) Let $A$ be $n \times m$ and $B$ be $m \times n$. Prove the Cauchy-Binet identity directly (i.e. using linear algebra):
$$\det(AB) = \sum_K \det(A_{[n],K}) \det(B_{K,[n]})$$
where $K \subset \{1, 2, \ldots, m\}$ varies over all subsets of size $n$.

(2) Use (a) to show that the set of TNN $n \times n$ matrices form a semigroup under matrix multiplication.

(3) Suppose $N$ is a planar network with $n$ sources and $m$ sinks, and let $N'$ be a planar network with $m$ sources and $n$ sinks. Let $A = M(N)$ and $B = M(N')$. Interpret the Cauchy-Binet identity in terms of non-intersecting paths. Can you deduce the Cauchy-Binet identity (for *all* matrices, not just TNN ones) in this way?

**Problem 2.** It may help to know Descartes' "rule of signs" for this problem.

(1) Suppose $0 < x_1 < x_2 < \cdots < x_n$. Prove that the $n \times n$ matrix $A = (a_{ij})$ with entries
$$a_{ij} = x_i^j$$
is TP.

(2) Suppose $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$. Prove that the $n \times n$ matrix $A = (a_{ij})$ with entries
$$a_{ij} = \exp(x_i y_j)$$
is TP. (Hint: argue that $\det(A)$ varies continuously with $x$ and $y$, and is never 0. Then reduce the problem to (a).)

**Problem 3.** Let
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$
be the $n$-th Catalan number. So $C_1 = 1, C_2 = 2, C_3 = 5, \ldots$. Prove that the infinite matrix
$$\begin{pmatrix} C_1 & C_2 & C_3 & \cdots \\ C_2 & C_3 & C_4 & \cdots \\ C_3 & C_4 & C_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$
is TP. What is the determinant of the top left $n \times n$ submatrix? (Hint: google "Dyck path".)

**Problem 4.**

(1) Let $A = (a_{ij})$ be a $k \times n$ matrix, where $k \leq n$. Denote the columns of $A$ by $c_1, c_2, \ldots, c_n \in \mathbb{C}^k$. For an ordered $k$-tuple $(j_1, j_2, \ldots, j_k) \subset \{1, 2, \ldots, n\}$ let $p_{j_1,j_2,\ldots,j_k}$ be the determinant of the $k \times k$ matrix with columns given by $c_{j_1}, c_{j_2}, \ldots, c_{j_k}$, in that order. Fix an integer $1 \leq r \leq k$. Prove the Plücker relation:

$$p_{i_1,i_2,\ldots,i_k} p_{j_1,j_2,\ldots,j_k} = \sum p_{i'_1,i'_2,\ldots,i'_k} p_{j'_1,j'_2,\ldots,j'_k}$$

where the summation is over all exchanges of $j_1, j_2, \ldots, j_r$ with $r$ of the indices amongst $i_1, i_2, \ldots, i_k$, maintaining the order in each. (Hint: prove that the difference of the two sides is a multilinear expression that is alternating in the $k+1$ vectors $c_{i_1}, c_{i_2}, \ldots, c_{i_k}, c_{j_r}$.) For example, if $I = \{1, 2\}$ and $J = \{3, 4\}$ and $r = 1$, we get

$$p_{12}p_{34} = p_{32}p_{14} + p_{13}p_{24} = -p_{23}p_{14} + p_{13}p_{24}.$$

(2) Calculate the dimension of the subspace $V_2 \subset \mathbb{C}[a_{ij}]$ spanned by the products $p_{i_1,i_2,\ldots,i_k} p_{j_1,j_2,\ldots,j_k}$ as $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ vary over all $k$-element subsets of $[n]$.

**Problem 5. (*)** Let $X$ be a $n \times n$ matrix of indeterminates $x_{ij}$. Let $V \subset \mathbb{C}[x_{ij}]$ be the subspace spanned by products of minors of the form

$$\det(X_{I,J}) \det(X_{\bar{I},\bar{J}})$$

where $\bar{I}$ denotes the complement of $I$ in $[n]$, and we have $|I| = |J|$. What is the dimension of $V$? Can you extend the answer to products of 3 of more minors?