Problem Set 2
Due on Wednesday Sept 25

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Problem 1. Prove that the following three sets are identical:

(1) $GL_n(\mathbb{R})_{>0}$
(2) $GL_n(\mathbb{R})_{\geq 0} \cap Bw_0B \cap B_-w_0B_-$ where w_0 denotes the longest element $n(n-1)\cdots 21$ of S_n, and B denotes the upper triangular matrices in GL_n.
(3) $U^-_{>0} \cdot T_{>0} \cdot U_{>0}$ where U^- are the lower triangular matrices with 1-s on the diagonal, and $T_{>0}$ is the set of diagonal matrices with positive diagonal entries.

Problem 2. Show that if $f(t)$ is a totally positive function, then so is $(f(-t))^{-1}$.

Problem 3. A polynomial function $p(x_{ij})$ in variables x_{ij} is called *totally nonnegative* if $p(X) \geq 0$ for any TNN matrix X.

(1) Let $w, v \in S_n$. Prove that

$$x_{1,w(1)} \cdots x_{n,w(n)} - x_{1,u(1)} \cdots x_{n,u(n)}$$

is TNN if $w \leq u$ in Bruhat order on S_n. (Hint: if $w < u$ in Bruhat order, then there is a chain $w = w_0 < w_1 < w_2 < \cdots < w_r = u$ where $w_i = w_{i+1}(ij)$; that is, successive permutations in the chain differ by a transposition.)
(2) **(*)** (This is not too hard.) Prove the converse of the previous statement.
(3) **(*)** The set of all totally nonnegative polynomials forms a cone: it is closed under addition, and multiplication by $\mathbb{R}_{>0}$. Compute this cone for 2×2 and 3×3 matrices.

Problem 4. A *complete matching* (just "matching" in this problem) on $[2n]$ is a set of edges in the complete graph K_{2n} with vertex set $[2n]$ which uses each vertex exactly once.

(1) Prove that the number of matchings on $[2n]$ is $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.
(2) Let π be a matching. The *crossing number* $c(\pi)$ of π is the number of (pairwise) intersections of edges when π is drawn in a disk, with the vertices arranged in circular order on the boundary of the disk. For a skew-symmetric matrix A, define the pfaffian

$$\mathrm{pf}(A) = \sum_{\pi} (-1)^{c(\pi)} \prod_{(i,j) \in \pi} a_{ij}$$

where the sum is over all matchings on $[2n]$, and in the product we always take $i < j$. For example for $n = 2$, we have $\mathrm{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

A proof of the following classical identity

$$\mathrm{pf}(A)^2 = \det(A)$$

can be found easily online.

Let N be a planar acyclic directed network as usual, with $2n$ sources and an arbitrary number of sinks. Define a skew-symmetric $2n \times 2n$ matrix $A(N)$ by setting

$$a_{ij} = \sum_{p,q} \mathrm{wt}(p)\mathrm{wt}(q)$$

for $i < j$, where the summation is over all pairs of noncrossing paths from sources i and j to any pair of sinks. Prove Stembridge's Pfaffian-analogue of the Lindström Lemma:

$$\mathrm{pf}(A(N)) = \sum_P \mathrm{wt}(P)$$

where the summation is over all noncrossing families of paths P using all the sources and any subset of sinks.

(3) Suppose $n = 2$ and N is a planar acyclic directed network with nonnegative edge weights and 4 sources. Let $A = A(N)$. Show that $a_{13}a_{24} - a_{14}a_{23} \geq 0$. Conclude that subpfaffian positivity is not enough to guarantee that a skew-symmetric matrix A is realizable by a network.

(4) **(*)** (This problem generalizes the previous one significantly.) Let $A = A(N)$ be a $2n \times 2n$ skew-symmetric matrix arising from a network N with nonnegative edge weights.
   (a) Suppose $|I| = |J|$ is even. Prove that $|A_{I,J}| \geq 0$.
   (b) Suppose $|I| = |J|$ is odd. Prove that $|A_{I,J}| \geq 0$ for all networks if and only if $i_1 \leq j_1, i_2, \leq j_2, \ldots$.

(5) **(*)** (Open?) Find semialgebraic conditions on a $2n \times 2n$ skew-symmetric matrix that guarantee realizability by a network. For example, are the conditions of the previous problem, together with nonnegativity of subpfaffians enough to guarantee realizability?

(6) **(*)** (Open?) Find "generators" for the set of $2n \times 2n$ skew-symmetric matrices that are realizable as $A(N)$ by a planar network.

Problem 5. (*) (This problem is not hard, just optional.) Suppose X is a $n \times n$ matrix. Fix $I, J \subset [n]$, $|I| = |J| = r$ and for $i \in [n]/I$, $j \in [n]/J$ let

$$y_{i,j} = |X_{I \cup i, J \cup j}|.$$

We assume the following basic determinantal identity (Sylvester's identity):

$$\det(Y) = |X_{I,J}|^{n-r-1}|X|.$$

(1) Let X be a $n \times (n+1)$ matrix. Fix integers $1 < k, \ell < n+1$. Use Sylvester's identity to prove

$$|X_{[n],[n+1]/l}||X_{[n]/k,[n]/1}| = |X_{[n],[n+1]/1}||X_{[n]/k,[n]/l}| + |X_{[n],[n]}||X_{[n]/k,[n+1]/\{1,l\}}|.$$

(Hint: apply Sylvester's identity with $r = n-1$ to the matrix obtained from X by adding a row with entries $(0, 0, \ldots, 0, 1)$.)

(2) Use (1) to prove the following Lemma due to Fekete. Assume X is an $n \times m$ matrix with $n \geq m$, such that the minors $|X_{I,[m-1]}|$ for any I are positive, and all minors of size m with consecutive (solid) rows are positive. Then all minors of X of size m are positive.

(3) Use (2) to obtain Fekete's criterion for total positivity: if all solid minors of X are positive, then X is TP.