Problem Set 3
Due on Friday October 4

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Problem 1. Prove that the collection of row-initial column-solid minors $|X_{[1,j-i+1],[i,j]}|$ with $1 < i \leq j \leq n$ parametrizes the TP part $U_{>0}$ of the upper unipotent subgroup U. (Hint: just do what we did in Lecture 2 of class. You are allowed and encouraged to be brief in your discussions. You may already have proved this in Pset 2, if so, just say so.)

Problem 2. Recall that we defined wiring diagrams of a reduced word in class. In class I suggested that the reduced word should be read right to left when we draw the wiring diagram left to right, but I think this is *wrong*. Read the reduced word from left to right, as you draw the wiring diagram from left to right.

We assume that wiring diagrams are embedded into a disk as usual. A *chamber* of the wiring diagram is a connected component of the complement of the wiring diagram in the disk. Label each wire by the index of the source vertex (that is, the one on the left), and label each chamber with the indices of the wires that are below this chamber. Call such a set a *chamber set*.

For example, the wiring diagram for $s_2 s_1 s_2 \in S_3$ has seven chambers, labeled $\emptyset, 1, 2, 3, 12, 23, 123$.

(1) Find a formula for the number of chambers of a wiring diagram (of a reduced word) in terms of the length $\ell(w)$.
(2) Prove that any chamber set S of a wiring diagram for a reduced word of w satisfies the condition: if $j \in S$ and $j < i$ and $w(j) < w(i)$ then $i \in S$.
(3) Prove the converse of the previous problem. Namely, if $S \subset [n]$ satisfies the stated condition, then it is a chamber set for some reduced word of w.
(4) Let $w = w_0$ be the longest permutation. Find a reduced word of w_0 so that the chamber sets are exactly the intervals $\{[i,j] \mid 1 \leq i \leq j \leq n\}$.

Problem 3. A *total positivity test* for $U_{>0}$ is a collection $\mathcal{C}$ of subsets of $[n]$, of size $\binom{n}{2}$ which is a test for a matrix $X \in U$ to lie in $U_{>0}$. More precisely, if $|X_{[1,|S|],S}| > 0$ for all $S \in \mathcal{C}$ then $X \in U_{>0}$. A *totally positive basis* for $U_{>0}$ is a collection $\mathcal{C}$ of subsets of $[n]$, of size $\binom{n}{2}$ such that every (not-identically vanishing on U) minor is a subtraction-free rational function in the minors inside $\mathcal{C}$. Clearly totally positive bases are total positivity tests.

(1) Prove that the collection of chamber sets for a reduced word of w_0 forms a total positivity test, after we remove the chamber sets $1, 12, 123, \ldots, [n]$. (Hint: First, prove this for a particular choice of reduced word corresponding to row-initial column-solid minors. Second, investigate what happens when we go from one reduced word to another by a single move $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.)
(2) Are these collections also totally positive bases for $U_{>0}$?
(3) **(*)** (Open?) Find all total positivity tests for $U_{>0}$.
(4) **(*)** (Open?) Find all totally positive bases for $U_{>0}$.

Problem 4. For a pair $I, J \subset [n]$ of subsets we write $I \prec J$ if $i < j$ for all $i \in I$ and $j \in J$. A pair $I, J \subset [n]$ of subsets is *strongly separated* if either $I - J \prec J - I$ or $J - I \prec I - J$. A collection $\mathcal{C} \subset 2^{[n]}$ is strongly separated if every pair of subsets in it is strongly separated.

(1) Prove that the collection of chamber sets for any reduced word of w_0 is strongly separated.

(2) **(*)** Prove that any maximal collection of strongly separated subsets of $[n]$ is the set of chamber sets for some reduced word of w_0.

Problem 5. **(*)** (This problem is highly recommended for those familiar with character theory of finite groups.) Let G be a finite group.

(1) Let V be a finite dimensional representation of G over $\mathbb{C}$. Choose a Hermitian inner product $(\cdot,\cdot)$ making V a unitary representation of G. Show that for any $v \in V$,
$$\phi(g) = (\rho(g) \cdot v, v)$$
is a positive definite function on G, as defined in class. Here ρ denotes the action of g on V.

(2) Conclude that every character of a representation of G is a positive-definite class function on G, and the same holds for any nonnegative real linear combination of characters.

(3) (Harder?) Conversely, suppose ϕ is a central, positive-definite function on G. Show that ϕ is a positive linear combination of irreducible characters of G. (Hint: here is one possible way. First prove that any positive-definite function on G is essentially the "square" of a function $f \in \mathbb{C}[G]$ under convolution. Then decompose f according to the decomposition of $\mathbb{C}[G]$ into isotypic components.)

Problem 6. **(*)** (This problem is likely to be most suitable for the term paper; it is a big subject.)

(1) Let $\chi : S_n \to \mathbb{C}$ be an irreducible character of S_n. Prove that the polynomial
$$\sum_{w \in S_n} \chi(w) x_{1w(1)} x_{2w(2)} \cdots x_{nw(n)}$$
is a totally nonnegative polynomial; that is, it takes nonnegative values on TNN matrices. For example, if χ is the sign character, we are considering the determinant. If χ is the trivial character, we are considering the permanent.

(2) (Open?) For a partition λ of n, define $\eta_\lambda : S_n \to \mathbb{C}$ by $\eta_\lambda = \sum a_{\lambda\mu} \chi^\mu$, where χ^μ are the irreducible characters, and $a_{\lambda\mu}$ is the coefficient of the Schur function s_μ in the monomial symmetric function m_λ. Prove, or disprove that the function
$$\sum_{w \in S_n} \eta_\lambda(w) x_{1w(1)} x_{2w(2)} \cdots x_{nw(n)}$$
is totally nonnegative.