Problem Set 4
Due on October 11

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

This problem set has a lot of symmetric functions, but most of them should be doable only using the properties of symmetric functions I mentioned in class. If you've forgotten some of this, look up Stanley's Enumerative Combinatorics Volume 2, Chapter 7.

Problem 1. Let

$$f(t) = e^{\gamma t} \prod_i \frac{1+\beta_i t}{1-\alpha_i t}$$

be a TP function, as in the Edrei-Thoma theorem. Let $\phi = \phi(f) : \mathrm{Sym} \to \mathbb{R}$ be the corresponding totally positive homomorphism of symmetric functions.

(1) Compute

$$\sum_{n \geq 0} \phi(e_n) t^n$$

where e_n denotes the n-th elementary symmetric function.

(2) Prove that

$$\phi(p_n) = \sum_i \alpha_i^n + (-1)^{n+1} \sum_i \beta_i^n$$

for $n \geq 2$.

(3) (If you are uncomfortable with characters of symmetric groups, skip this.) Explain how the previous problem gives the value of the indecomposable character χ (corresponding to ϕ) at a cycle of length n.

(4) (*) Find a formula for $\phi(s_\lambda)$ as the generating function of certain tableaux.

Problem 2. (This problem is intended only for those who aren't familiar with symmetric functions. If you are, skip this.) Recall that we defined the Schur function as a determinant in homogeneous symmetric functions. Prove, using the same technique as the Lindstr̈om Lemma, the monomial expansion

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

of Schur functions. Here $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and weight μ, as described in class. (Hint: interpret semistandard Young tableaux in terms of non-intersecting lattice paths. If you are stuck, look at p.343 of Enumerative Combinatorics Vol 2.)

Problem 3. Let $\mathbb{Y}$ denote Young's graph of partitions that we defined in class, with edges given by partitions differing by one box. For a partition λ, let $d(\lambda)$ be the number of paths from the empty partition $\emptyset$ to λ in $\mathbb{Y}$, where we only consider *increasing* paths: that is, paths that always add boxes. (Though it is not relevant to this problem, the number $d(\lambda)$ is equal to the dimension of the irreducible representation of S_n labeled by λ.)

(1) Suppose $\phi : \mathbb{Y} \to \mathbb{R}_{\geq 0}$ is a normalized harmonic function. Let $\psi : \mathbb{Y} \to \mathbb{R}_{\geq 0}$ be defined by

$$\psi(\lambda) = \phi(\lambda) d(\lambda).$$

Prove that ψ restricted to the n-th level $\mathbb{Y}_n$ (consisting of partitions of n), gives a probability measure on $\mathbb{Y}_n$.

(2) Given ϕ and ψ related as in (2), let us define a transient Markov chain on $\mathbb{Y}$ as follows. Suppose μ is obtained from λ by adding a box. Then define the transition probability
$$p(\lambda, \mu) = \phi(\mu)/\phi(\lambda).$$
Let $\nu^{(0)} = \emptyset, \nu^{(1)}, \nu^{(2)}, \ldots$ be the random path in $\mathbb{Y}$, starting at the empty partition, given by these transition probabilities. Prove that
$$\mathrm{Prob}(\nu^{(n)} = \lambda) = \psi(\lambda)$$
for a partition λ of n. Prove also that the probability that the first n steps of the random path equals a particular n-step increasing path depends only on the endpoint of the path, and not the path itself.

Problem 4. (*) (If you are familiar with skew Schur functions, try to avoid explicitly using them for this problem.)

Consider the infinite matrix
$$X = \begin{pmatrix} 1 & h_1 & h_2 & h_3 & \cdots \\ 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 1 & h_1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$
with matrix entries lying in the ring of symmetric functions. Show that every minor of X is a nonnegative linear combination of Schur functions.

Problem 5. (*) Recall that the set of TP functions can be identified with the set of algebra homomorphisms $\phi : \mathrm{Sym} \to \mathbb{R}$ satisfying $\phi(s_\lambda) \geq 0$. Classify the algebra homomorphisms $\phi : \mathrm{Sym} \to \mathbb{R}$ satisfying $\phi(m_\lambda) \geq 0$, where m_λ denotes the monomial symmetric functions.

In other words, find and prove and analogue of the Edrei-Thoma theorem with monomial symmetric functions replacing Schur functions.

Problem 6. (*) Fix a positive integer n, and consider the set $T(n)$ of finite upper-triangular Toeplitz matrices:
$$X = \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
for $n = 4$.

(1) Prove that $T(n) \cap B_- w B_-$ is nonempty for exactly 2^{n-1} choices of $w \in S_n$.

(2) (I don't know an easy proof of this result, but there deserves to be one!) Let $\Delta_i : T(n) \to \mathbb{R}$ denote the upper-right justified $i \times i$ minor. Prove that that the map $\Delta : T(n)_{\geq 0} \to \mathbb{R}^{n-1}_{\geq 0}$
$$X \longmapsto (\Delta_1(X), \Delta_2(X), \ldots, \Delta_{n-1}(X))$$
is a bijection (in fact, a homeomorphism). For example, for $n = 4$, the matrix X above would be taken to $(a_3, a_2^2 - a_1 a_3, a_1^3 - 2a_1 a_2 + a_3)$.