Problem Set 5
Due on October 25

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Please also submit a starred problem from this, or a previous problem set.

Problem 1. Let $a, b \in \mathbb{R}_{>0}$. Prove that the matrix

$$\begin{pmatrix} \cosh(\sqrt{abt}) & \sqrt{a/bt}\,\sinh(\sqrt{abt}) \\ \sqrt{bt/a}\,\sinh(\sqrt{abt}) & \cosh(\sqrt{abt}) \end{pmatrix}$$

lies in $GL_2(\mathbb{R}((t)))_{\geq 0}$. (Hint: this matrix is an analogue of the exponentials $e^{\gamma t}$ in the factorization of TP functions.)

Problem 2. Suppose $A(t) \in GL_n(\mathbb{R}((t)))$. Show that the rows of the matrix X_A, considered as vectors in $\mathbb{R}^\infty$, are linearly independent.

Problem 3. Let N be a cylindric network (acyclic, directed from one boundary to the other, weighted). Let $X = M(N)$ (the path generating function matrix of N) have folding $A(t)$. Show (using network combinatorics) that the $k \times k$ minors of $A(t)$ have nonnegative coefficients when k is odd, and have sign-alternating coefficients when k is even. For example, when $k = 1$, the claim is that all the coefficients of $a_{ij}(t)$ are nonnegative.

Problem 4. Fix $n > 1$.

(1) What is the determinant of the (folded versions of) curl matrices $N(a_1, \ldots, a_n)$ and whirl matrices $M(a_1, \ldots, a_n)$?

(2) Let $A(t) \in \tilde{U}_{\geq 0} \subset GL_n(\mathbb{R}((t)))_{\geq 0}$. Suppose in the factorization theorem we have

$$A(t) = \left(\prod_{i=1}^{\infty} N(\mathbf{a}^{(i)})\right) Y(t) \left(\prod_{i=-\infty}^{-1} M(\mathbf{a}^{(i)})\right).$$

What is the relationship between $\det A(t)$ and $\det Y(t)$?

(3) Suppose $n = 2$. Prove that $\det Y(t)$ in the previous theorem is equal to $e^{\gamma t}$. (Hint: let $B(t) = A(t)^{-1}$. Then compare the Edrei-Thoma factorization of $b_{11}(t)$ with that of $a_{22}(t)$.)

(4) Now allow n to be arbitrary. As we remarked in class, each entry $a_{ij}(t)$ is a TP function and thus,

$$a_{ij}(t) = e^{\gamma t} \frac{\prod_{i=1}^{\infty}(1 + \beta_i t)}{\prod_{i=1}^{\infty}(1 - \alpha_i t)}.$$

Prove that if $n > 1$ then $\gamma = 0$. (Hint: first reduce to $n = 2$. Then use the previous problem.)

(5) Prove that the function $\det(Y(t))$ from (2) is equal to the function 1.

(6) **(*)** (Open?) Let $A(t) \in \tilde{U}_{\geq 0}$. Then each $a_{ij}(t)$ is a TP function with no "exponential part". What is the relationship between the poles and zeroes of different $a_{ij}(t)$?

Problem 5. (*) This problem is about the generators $x_i(a) \in U_{\geq 0}$. Recall (and check if you never did!) the relation:

$$x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}(bc/(a+c))x_i(a+c)x_{i+1}(ab/(a+c)).$$

As shorthand we may write the above relation as $(i, i+1, i) \leftrightarrow (i+1, i, i+1)$. There is also a commutation relation corresponding to $(i, j) \leftrightarrow (j, i)$ for $|i-j| > 1$.

(1) Due to laziness, commas are omitted in the following. Prove that the sequence of relations

$$(121321) \leftrightarrow (212321) \leftrightarrow (213231) \leftrightarrow (231213) \leftrightarrow$$
$$(232123) \leftrightarrow (323123) \leftrightarrow (321323) \leftrightarrow (321232) \leftrightarrow (312132) \leftrightarrow (132312)$$
$$(123212) \leftrightarrow (123121) \leftrightarrow (121321)$$

changes the parameters a, b, c, d, e, f in $x_1(a)x_2(b)x_1(c)x_3(d)x_2(e)x_1(f)$ back to themselves. (Hint: given what we have proved, this is supposed to be very easy.)

(2) Two reduced words for $w \in S_n$ are commutation equivalent if they are related by the moves $ij \sim ji$ for $|i-j| > 1$. Let G_w be the graph on commutation equivalent classes of reduced words of w, with edges whenever two (representatives of commutation classes of) reduced words are related by a move $i(i+1)i \sim (i+1)i(i+1)$.

Prove that the fundamental group of G_w is generated by 4-cycles of the form

$$i, i+1, i \cdots j, j+1, j \sim i+1, i, i+1 \cdots j, j+1, j \sim$$
$$i+1, i, i+1 \cdots j+1, j, j+1 \sim i, i+1, i, j+1, j, j+1 \sim i, i+1, i \cdots j, j+1, j$$

and an 8-cycle corresponding to the sequence of moves in the previous problem. (The sequence in the previous part looks longer than an 8-cycle, but after one removes the commutation moves, it becomes an 8-cycle).

Problem 6. (*) (Open?) Let $X = \prod_{i=-\infty}^{\infty} N(\mathbf{a}^{(i)})$ be a bi-infinite product of curls such that the sum of all the parameters is bounded. Factor X into canonical form $X = AYB$ where A is a product of curls, B is a product of whirls, and Y is doubly entire.