## Problem Set 6
Due on November 6

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Please also start to think about what you would like to do for your final paper. You can read and write about any topic that we touched upon but did not cover in depth, or you could submit some partial progress on one of the open starred problems. I'd like you to discuss your preliminary plan with me by November 15.

Problem 1. A one-row tableau is just the same as a weakly-increasing sequence of positive integers. Suppose we are given two one-row tableaux T, S. *Jeu-de-taquin* is the following algorithm: place the two tableaux diagonally relative to each other, and start sliding the boxes in until you get a two-row tableaux, always using the rules that columns are strictly increasing, and rows are weakly increasing. For $T = 123$ and $S = 11333$ we would have
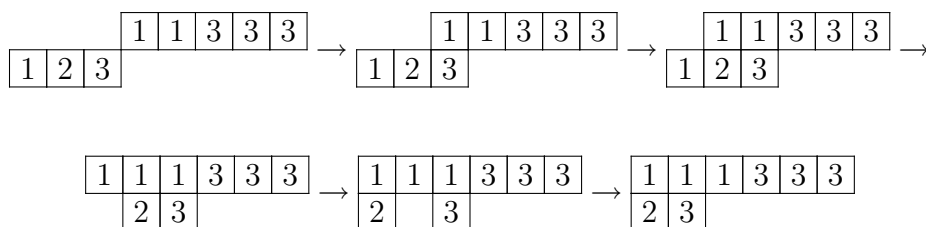
So $jdt(T, S)$ is the two-row tableaux displayed at the end. Note that in the first few slides it looks like I slid the entire first row in each move, but actually I should be sliding one box at a time, and checking the inequalities each step. (It would be a much longer example, if I did it one box at a time, like I did at the end.)

(1) If you have never seen this before, convince yourself that this is well-defined: you'll never get stuck, and you'll never have multiple choices that lead to different answers.

(2) Let T have length m and S have length n. Prove that there exists unique one-row tableaux U of length n and V of length m such that $jdt(T, S) = jdt(U, V)$. We write $\tilde{R}(T, S) = (U, V)$, and call $\tilde{R}$ the *combinatorial R-matrix*. In the above example, we have $\tilde{R}(T, S) = (12333, 113)$.

If you want you only need to **prove this for tableaux with only the numbers 1 and 2**. It is already interesting there. Those of you who know about symmetric functions can probably easily prove the statement in general.

For those who know about $GL(V)$-rep theory, the corresponding representation theoretic statement is: the representation $\text{Sym}^m(V) \otimes \text{Sym}^n(V)$ is multiplicity-free when written as a direct sum of irreducibles, and is isomorphic to $\text{Sym}^n(V) \otimes \text{Sym}^m(V)$.

(3) Observe that $\tilde{R}$ is an involution, and that for each $i > 0$ it preserves the total number of i-s in the two tableaux.

(4) (*) Prove that $\tilde{R}$ satisfies the braid relation when applied to three one-row tableaux (T, S, U). That is, $\tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12} = \tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{12}$, where $\tilde{R}_{12}$ is the R-matrix applied to the first two of three tableaux. (From the representation theory point of view, this relation is at the origin of quantum groups.)

Problem 2. Fix $n \geq 2$. Recall that we have defined the (geometric) R-matrix as follows. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k$$

and set $R(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ where

$$b_i' = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a_i' = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

(1) Let $r(\mathbf{a}) = \prod_{i=1}^{n} a_i$. Show that $r(\mathbf{a}') = r(\mathbf{a})$ and $r(\mathbf{b}') = r(\mathbf{b})$. Show also that $b_i' a_{i+1}' = a_i b_{i+1}$.

(2) In class we showed that $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$. Prove that the R-matrix satisfies the braid relation when applied to $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. (Hint: First argue that the braid relation is an identity of rational functions, so it suffices to check it for say $r(\mathbf{a}) > r(\mathbf{b}) > r(\mathbf{c})$. Now use the lemma from class that if $X \in \tilde{U}_{\geq 0}$ is not finitely supported and

$$X' = M(-a_1, \ldots, -a_n)X \in \tilde{U}_{\geq 0}$$

then $a_i \leq \epsilon_i(X)$ for all i.)

Problem 3. If $f(x, y, z, \ldots)$ is a subtraction-free rational function, we denote by $\mathrm{trop}(f)$ the piecewise-linear function in the same variables, obtained by replacing $+$ by $\min$, and $\times$ by $+$, and $/$ by $-$. For example, if $f = (xy + z)/(x^2 + y)$ then $\mathrm{trop}(f) = \min(x + y, z) - \min(2x, y)$.

(1) Check that $\mathrm{trop}(f)$ is well-defined: if $f = g$ are two subtraction-free rational expressions for the same rational function, then $\mathrm{trop}(f)$ and $\mathrm{trop}(g)$ are two expressions for the same piecewise-linear function.

(2) Prove that $\mathrm{trop}R : (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{b}', \mathbf{a}')$ maps $\mathbb{Z}_{\geq 0}^{2n}$ to itself.

(3) For $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ write $\bar{\mathbf{a}} = (a_n, a_1, \ldots, a_{n-1})$. Suppose T, S are two one-rowed tableaux using the numbers $1, 2, \ldots, n$. Let $\tilde{R}(T, S) = (U, V)$ and let a_i, b_i, c_i, d_i be the number of i-s in T, S, U, V respectively.

   (a) Prove that

$$(\mathrm{trop}R)(\mathbf{a}, \bar{\mathbf{b}}) = (\mathbf{c}, \bar{\mathbf{d}})$$

     for $n = 2$.

   (b) **(*)** Prove the same relation for $n > 2$.

   For the tableaux above, we have $\mathbf{a} = (1, 1, 1)$, $\bar{\mathbf{b}} = (3, 2, 0)$, $\mathbf{c} = (1, 1, 3)$, and $\bar{\mathbf{d}} = (1, 2, 0)$. Then for example

$$1 = c_2 = \mathrm{trop}(\frac{\bar{b}_3 \kappa_3}{\kappa_2}) = 0 + \min(2, 4, 5) - \min(2, 1, 3).$$

(4) Interpret the tropicalization of Problem 2 (1) in terms of tableaux.

Problem 4.

(1) Let $n = 3$. Prove that there are braid limits from $(1012)^{\infty}$ to both $(012)^{\infty}$ and $(102)^{\infty}$.

(2) Let $n = 3$. Suppose $\mathbf{i}$ is an infinite reduced word, such that no braid moves can be applied. Prove that $\mathbf{i} = c^{\infty}$ where c is a reduced word of some Coxeter element. We call c^{∞} an infinite Coxeter word.

(3) Now let n be arbitary. An infinite reduced word $\mathbf{i}$ is *fully commutative* if for any $\mathbf{j}$ related to $\mathbf{i}$ be finitely many braid moves, the only braid moves that you can apply to $\mathbf{j}$ are the commutation moves $ij = ji$ for $|i - j| \geq 2$. Prove that an infinite reduced word is fully commutative if and only if it is braid equivalent to an infinite Coxeter word.

(4) (*) Let $\mathbf{i}$ be any infinite reduced word. Prove that there is a braid limit $\mathbf{i} \to c^\infty$, for an infinite Coxeter word c^∞.

Problem 5. (*) Call an infinite reduced word $\mathbf{i}$ *injective* if the map $x_\mathbf{i} : \ell_{>0}^1 \to \tilde{U}_{\geq 0}$ is injective.

(1) Prove that all infinite Coxeter words for $n = 3$ are injective.

(2) (Open?) Is the same true for $n > 3$?

Problem 6. (*) (**Open?**) The following *Principal Ideal Conjecture* seems to be the most attractive conjecture in the theory of the maps $x_\mathbf{i}$. Let $X \in \Omega = \cup_\mathbf{i} E_\mathbf{i}$. Let $I(X)$ be the set of braid equivalence classes $[\mathbf{i}]$ of infinite reduced words such that $X \in E_\mathbf{i}$.

By the theorem from class, $I(X)$ is a lower order ideal: if $[\mathbf{i}] \in I(X)$ and $[\mathbf{j}] \leq [\mathbf{i}]$ then $[\mathbf{j}] \in I(X)$. Is $I(X)$ a principal ideal in limit weak order? That is, is $I(X) = \{[\mathbf{j}] \mid [\mathbf{j}] \leq [\mathbf{i}_X]\}$ for some $\mathbf{i}_X$?

Problem 7. (*) Let $\mathbb{R}(t)$ denote the field of rational functions $p(t)/q(t)$ with real coefficients. Let $\tilde{U}(\mathbb{R}(t)) \subset \tilde{U}$ be the subgroup of the unipotent part of the formal loop group such that every entry can be written as a rational function. Prove that $\tilde{U}(\mathbb{R}(t))_{\geq 0}$ is generated by $M(a_1, a_2, \ldots, a_n)$, $N(b_1, b_2, \ldots, b_n)$ and $x_i(c)$, where all parameters are nonnegative.