NOTES ON THE TOTALLY NONNEGATIVE GRASSMANNIAN

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1. Introductory comments

The theory of the totally nonnegative part of the Grassmannian was introduced by Postnikov around a decade ago. The subject has become one of the most active areas in algebraic combinatorics. These notes give a condensed (and not complete) introduction to this subject and was written as part of a course on total positivity. The aim was to obtain the main statements (Theorem 3.10) as quickly as possible, without introducing all the different aspects of the theory.

Let us make some bibliographical comments. The main difference to Postnikov's theory is that we have chosen to use perfect matchings instead of path counting to define boundary measurements. The possibility of this was observed by Postnikov, Speyer and Williams [PSW] following work of Talaska [Tal]. That these boundary measurements satisfy the Plücker relations is a theorem of Kuo [Kuo]. We have also borrowed some language (such as that of connections on line bundles on graphs) from Goncharov and Kenyon [GK].

One drawback of this approach is that we use only planar bipartite graphs instead of Postnikov's more general plabic graphs. We have also chosen to use the bounded affine permutations of [KLS] instead of Postnikov's decorated permutations.

A possible novelty is that we define (Proposition 2.3) some Grassmannian analogues of Rhoades and Skandera's Temperley-Lieb immanants. This leads to some inequalities (Section 3.5) between products of boundary measurements, suggesting a log-concavity property reminiscient of results in Schur positivity [LPP]. One consequence of this approach is an easy proof that positroids are sort-closed, which combined with the general theory of alcoved polytopes [LP], gives a new proof of Oh's theorem [Oh] that positroids are intersections of cyclically rotated Schubert matroids. These ideas are also closely related to work of Kenyon and Wilson [KW].

Our proof that every point in the TNN Grassmannian is representable by a network does not seem to have appeared before in the literature – it somewhat simplifies, in our opinion, the part of the subject that relies on Le-diagrams. I learnt the idea of using bridge decompositions and lollipops to generate all planar bipartite graphs from Postnikov.

A very important theorem (Theorem 2.17) stated but not proved in these notes is that reduced planar bipartite graphs with the same bounded affine permutation are connected by moves.

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2. Boundary measurements of planar bipartite graphs

We will try to follow the convention: N will denote a weighted network, and G will denote an unweighted graph.

2.1. **Matchings for bipartite graphs in a disk.** Let N be a weighted bipartite network embedded in the disk with n boundary vertices, labeled $1, 2, \ldots, n$ in clockwise order. Each vertex (including boundary vertices) is colored either black or white, and all edges join black vertices to white vertices. We let d be the number of interior white vertices minus the the number of interior white vertices. Furthermore we let $d' \in [n]$ be the number of white boundary vertices. Finally, we assume that all boundary vertices have degree 1, and that edges cannot join boundary vertices to boundary vertices.

Remark 2.1. *Since the graph is bipartite, this last condition ensures that the coloring of the boundary vertices is determined by the interior part of the graph. So sometimes we will pretend that boundary vertices are not colored.*

An **almost perfect matching** Π is a subset of edges of N such that

(1) each interior vertex is used exactly once
(2) boundary vertices may or may not be used.

The boundary subset $I(\Pi) \subset \{1, 2, \ldots, n\}$ is the set of black boundary vertices that are used by Π union the set of white boundary vertices that are not used. By our assumptions we have $|I(\Pi)| = k := d' + d$.

Remark 2.2. *We will always assume that almost perfect matchings of N do exist. Therefore, we may suppose that isolated interior vertices do not exist.*

Define the **boundary measurement**, or **dimer partition function** as follows. For $I \subset [n]$ a k-element subset,

$$\Delta_I(N) = \sum_{\Pi : I(\Pi) = I} \mathrm{wt}(\Pi)$$

where $\mathrm{wt}(\Pi)$ is the product of the weight of the edges in Π.

A **partial non-crossing matching** τ is a matching of a subset $I(\tau) \subset \{1, 2, \ldots, n\}$ of even size, such that when the vertices are arranged in order on a circle, and the edges are drawn in the interior, then the edges do not intersect.

Let Π and Π' be two almost perfect matchings of a network N. Then the **double matching** $\Sigma = \Pi \cup \Pi'$ is a union of doubled edges, interior cycles, and paths between boundary vertices. The set S of vertices used by the paths on the double matching is given by $S = (I(\Pi) \setminus I(\Pi')) \cup (I(\Pi') \setminus I(\Pi))$. Thus each double matching Σ gives rise to a partial non-crossing matching on $S \subset \{1, 2, \ldots, n\}$. Note that a double matching can arise from a pair of matchings in many different ways.

For each partial non-crossing matching τ, and each subset of T of $[n] \setminus I(\tau)$, let

$$\Delta_{\tau,T} = \sum_{\Sigma} \mathrm{wt}(\Sigma)$$

be the sum over double matchings which give boundary matching τ, and T contains white boundary vertices used twice in Σ, together with black boudnary vertices not used in Σ. Here $\mathrm{wt}(\Sigma)$ is the product of all weights of edges in T times $2^{\#cycles}$.

Given $I, J \in \binom{[n]}{k}$, we say that a partial non-crossing matching τ is **compatible** with I, J if $I(\tau) = (I \setminus J) \cup (J \setminus I)$, and each edge of τ matches a vertex in $(I \setminus J)$ with a vertex in $(J \setminus I)$.

Proposition 2.3. *We have*

$$\Delta_I(N)\Delta_J(N) = \sum_{\tau,T} \Delta_{\tau,T}$$

where the summation is over all partial non-crossing matchings τ compatible with I, J, and $T = I \cap J$.

Proof. The only thing left to prove is the compatibility property.

Let Π, Π' be almost perfect matchings of N such that $I(\Pi) = I$ and $I(\Pi') = J$. Let p be one of the boundary paths in $\Pi \cup \Pi'$, with endpoints s and t. If s and t have the same color, then the path is even in length. If s and t have different colors, then the path is odd in length. In both cases one of s and t belongs to $I \setminus J$ and the other belongs to $J \setminus I$. ☐

Theorem 2.4. *Suppose N has nonnegative real weights, and that almost perfect matchings of N exist. Then the homogeneous coordinates $(\Delta_I(N))_{I \in \binom{[n]}{k}})$ defines a point $M(N)$ in the Grassmannian $\mathrm{Gr}(k,n)$.*

We shall use the following result.

Proposition 2.5. *A non-zero vector $(\Delta_I(N))_{I \in \binom{[n]}{k}})$ lies in $\mathrm{Gr}(k,n)$ if and only if the Plücker relation with 1 index swapped is satisfied:*

$$\sum_{r=1}^{k} \Delta_{i_1,i_2,\ldots,i_{k-1},j_r} \Delta_{j_1,\ldots,j_{r-1},\hat{j}_r,j_{r+1},\ldots,j_k} = 0 \tag{1}$$

where $\hat{j}_r$ denotes omission.

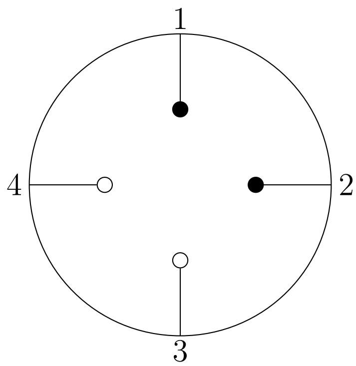
The convention is that Δ_I is antisymmetric in its indices, so for example $\Delta_{13} = -\Delta_{31}$.

Proof of Theorem 2.4. Use Proposition 2.3 to expand (1) as a sum of $\Delta_{\tau,T}$ over pairs (τ, T) (with multiplicity). We note that the set T is always the same in any term that comes up. We assume that $i_1 < i_2 < \cdots < i_{k-1}$ and $j_1 < j_2 < \cdots < j_{k+1}$.

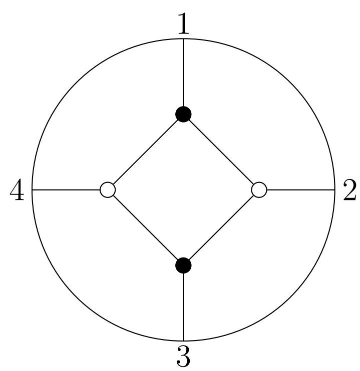
So each term $\Delta_{\tau,T}$ is labeled by (I, J, τ) where I, J is compatible with τ, and I, J occur as a term in (1). We provide an involution on such terms. By the compatibility condition, all but one of the edges in τ uses a vertex in $\{i_1, i_2, \ldots, i_{k-1}\}$. The last edge is of the form (j_a, j_b), where $j_a \in I$ and $j_b \in J$. The involution swaps j_a and j_b in I, J but keeps τ the same.

Finally we show that this involution is sign-reversing. Let $I' = I \cup \{j_b\} - \{j_a\}$ and $J' = J \cup \{j_a\} - \{j_b\}$. Then the sign associated to the term labeled by (I, J, τ) is equal to (-1) to the power of $\#\{r \in [k] \mid i_r > j_a\} + a$. Note that by the non-crossingness of the edges in τ there must be an even number of vertices belonging to $(I \setminus J) \cup (J \setminus I)$ strictly between j_a and j_b. Thus $j_b - j_a = (b-a) + (\#\{r \in [k] \mid i_r > j_b\} - \#\{r \in [k] \mid i_r > j_a\})$ mod 2 is odd. So the sign changes. $\square$

Example 2.6. *Let us consider a lollipop graph N. Note that all boundary vertices must have degree 1, so we can't have graphs smaller than the lollipop graphs. The point $M(N) \in \mathrm{Gr}(k,n)$ is a torus-fixed point.*



Example 2.7. *Le us compute the boundary measurments of the square graph for*



$\mathrm{Gr}(2,4)$.

In fact, the formula in Proposition 2.3 can be explicitly inverted when an explicit subset of $\{\Delta_I \Delta_J\}$ is chosen. We say that $\Delta_I \Delta_J$ is a **standard monomial** if $i_r \leq j_r$ for all r (in other words, I, J form the columns of a semistandard tableau).

Proposition 2.8. *There is a bijection θ between standard monomials and pairs (τ, T), and a partial order $\leq$ on standard monomials such that the transition matrix between $\{\Delta_I \Delta_J \mid (I,J) \text{standard}\}$ and $\{\Delta_{\tau,T}\}$ is invertible and triangular. More precisely,*

$$\Delta_I \Delta_J = \sum_{(I',J') \leq (I,J)} \Delta_{\theta(I',J')}.$$

In particular, $\{\Delta_{\tau,T}\}$ forms a basis for the degree two part of the homogeneous coordinate ring of the $\mathrm{Gr}(k,n)$ in the Plücker embedding.

Proof. Since the subset $T = I \cap J$ plays little role, we shall assume $T = \emptyset$, and for simplicity, $I \cup J = [n]$.

Then (I,J) is a two-row tableaux using the number $1, 2, \ldots, 2k = n$. The bijection θ sends such I, J to the non-crossing matching τ on $[2k]$ given by connecting i_r to j_s

where s is chosen minimal so that $\#(I \cap (j_s - i_r)) = \#(J \cap (j_s - i_r))$. This bijection can be described in terms of Dyck paths as follows: draw a Dyck path having a diagonally upward edge E_i at positions specified by $i \in I$ and a diagonally downward edge D_j at positions specified by $j \in J$. Then τ joins i to j if the horizontal rightwards ray starting at E_i intersects D_j before it intersects any other edge. □

2.2. **Gauge equivalence.** If $e_1, e_2, \ldots, e_d$ are adjacent to an **interior** vertex v, we can multiply all of their edge weights by the same constant $c \in \mathbb{R}_{>0}$, and still get the same point $M(N)$. Note that we can't do this at a boundary vertex.

Let F be any face of the graph N. This can be a face completely bounded by edges of N, or faces that also touch the boundary of the disk. Take the clockwise orientation of the edges bounding the face, and define the face weight

$$y_F = \mathrm{wt}(F) = \prod_{e \text{ bounding } F} \mathrm{wt}(e)^{\pm 1}$$

where we have $+1$ if the edge goes out of a black vertex and into a white vertex, and -1 if the edge goes out of a white vertex and into a black vertex.

Lemma 2.9. *Face weights are preserved by gauge equivalence.*

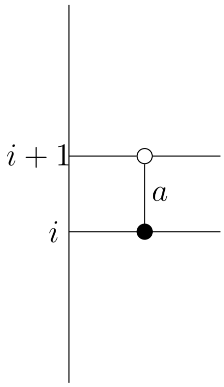
Here is some more abstract language to formulate the above. A **line bundle** $V = V_G$ on a graph G is the association of a one-dimensional vector space V_v to each vertex v of G. A connection Φ on V is a collection of invertible linear maps $\phi_{uv} : V_u \to V_v$ for each edges u, v satisfying $\phi_{uv} = \phi_{vu}^{-1}$. If we fix a basis of each V_v, then the connection Φ is equivalent to giving G a weighting, that is, it is equivalent to a weighted network N with underlying graph G.

Lemma 2.10. *Gauge equivalence for N corresponds to changing bases for $\{V_v\}$. Connections on V are in bijection with gauge equivalence classes of weighted networks N with underlying graph G. Connections are in bijection with face weights y_F, which can be chosen arbitrarily subject to the condition that $\prod_F y_F = 1$.*

Proof. Only the last statement is not clear, and it basically follows from Euler's formula. □

Let $\mathcal{L}_G$ be the moduli space of connections on V_G, and let $(\mathcal{L}_G)_{>0}$ be the positive points so that $(\mathcal{L}_G)_{>0} \simeq \mathbb{R}_{>0}^{\#F-1}$ can be identified with the space of positive real weighted networks with underlying graph G, modulo gauge equivalence. Here $\#F$ denotes the number of faces of G.

2.3. **Generators.** By adding degree two vertices, we can always assume that a boundary vertex is the color we want it to be. If i and $i+1$ are two adjacent boundary vertices, we can add a **bridge** between the two edges leaving i and $i+1$. There are two different kinds of bridges depending on which color is assigned to which vertex of the added edge. For simplicity, we for example just say we are adding "a bridge with white at $i+1$ and black at i".



These are our analogues of the Chevalley generators $x_i(a)$ and $y_i(b)$.

Lemma 2.11. *Let N be a network. Now let N' be obtained by adding a bridge with edge weight a from i to $i+1$ which is white at i and black at $i+1$. Then the boundary measurements change as follows:*

$$\Delta_I(N') = \begin{cases} \Delta_I(N) + a\Delta_{I-\{i+1\}\cup\{i\}}(N) & \text{if } i+1 \in I \text{ but } i \notin I \\ \Delta_I(N) & \text{otherwise.} \end{cases}$$

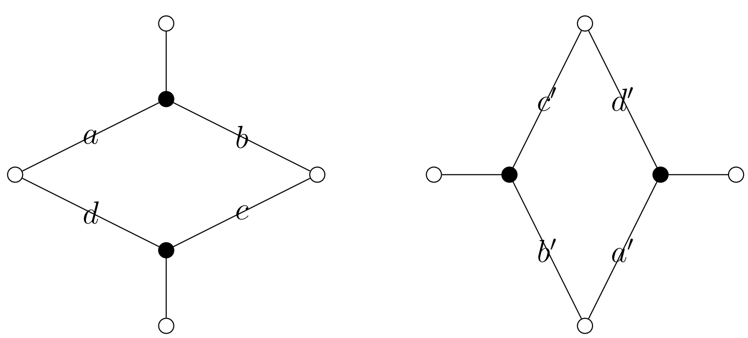
If the bridge is black at i and white at $i+1$, then

$$\Delta_I(N') = \begin{cases} \Delta_I(N) + a\Delta_{I-\{i\}\cup\{i+1\}}(N) & \text{if } i \in I \text{ but } i+1 \notin I \\ \Delta_I(N) & \text{otherwise.} \end{cases}$$

2.4. **Relations for bipartite graphs.** We have the following relations:

(M1) Spider move or square move: assuming the leaf edges of the spider have been gauge fixed to 1, the transformation is

$$a' = \frac{a}{ac+bd} \qquad b' = \frac{b}{ac+bd} \qquad c' = \frac{c}{ac+bd} \qquad d' = \frac{d}{ac+bd}$$



(M2) Valent two vertex removal. If v has degree two, we can gauge fix both incident edges (v, u) and (v, u') to have weight 1, then contract both edges (that is, we remove both edges, and identify u with u'). Note that if v is a valent two-vertex adjacent to boundary vertex b, with edges (v, b) and (v, u), then removing v produces an edge (b, u), and the color of b flips.

(R1) Multiple edges with same endpoints is the same as one edge with sum of weights.

(R2) Leaf removal. Suppose v is leaf, and (v, u) the unique edge adjacent to it. Then we can remove both v and u, and all edges adjacent to u. However, if there is a boundary edge (b, u) where b is a boundary vertex, then that edge is replaced by a boundary edge (b, w) where w is a new vertex with the same color as v.
(R3) Dipoles (two degree one vertices joined by an edge) can be removed.

Remark 2.12. *If after a move the condition that boundary vertices are degree 1 fails to hold, then we can use move (M2) before hand. One consequence is that leaves joined to a boundary vertex cannot actually be removed.*

By (R2) we can always remove leaves that are not adjacent to the boundary. Leaves connected to the boundary are called **boundary leaves**. We will call G **leafless** if the only leaves are boundary leaves.

Proposition 2.13. *Each of these relations preserves $M(N)$.*

The relations generate the **move-equivalence class** of N.

Proposition 2.14. *The relations for N imply the braid relations for Chevalley generators $x_i(a)$.*

2.5. **Zig-zag paths and trips.** Let G be the underlying unweighted graph of N. In the following we will sometimes think of an edge in G as two directed edges, one in each direction.

We decompose the underlying graph G of N into directed paths and cycles as follows. Given a directed edge $e : u \to v$, if v is black we pick the edge $e' : v \to w$ after e by turning (maximally) right at v; if v is white, we turn (maximally) left at v. This decomposes G into a union of directed paths and cycles, such that every edge is covered twice (once in each direction). These paths and cycles are called zig-zag paths, or trips.

Zig-zag paths, or trips, can also be drawn as **strands** in the medial graph of G. The medial graph Γ of G has a 4-valent vertex for each edge of G, and additionally has two vertices on the boundary of the disk between each pair of adjacent boundary vertices of G. We have an edge in Γ between two vertices whenever the corresponding edges of G belong to the same face. The graph Γ can be decomposed into paths that go straight through each 4-valent vertex, and in addition is directed in the same way as the corresponding zig-zag path.

The **trip permutation** $\pi_G : [n] \to [n]$ is the permutation given by $\pi_G(i) = j$ if the trip that starts at i ends at j.

Proposition 2.15. *Trip permutations are preserved by the moves (M1) and (M2).*

Proof. This is checked case by case. □

A leafless bipartite graph G is **reduced** or **minimal** if

(1) there are no trips that are cycles
(2) no trip uses an edge twice (once in each direction) except for the case of a boundary leaf
(3) no two trips T_1 and T_2 share two edges e_1, e_2 such that the edges appear in the same order in both trips

Note that T_1 and T_2 can use the same edge if they appear in a different order.

Remark 2.16. *The conditions imply that if $\pi_G(i) = i$ then the boundary vertex i must be connected to a boundary leaf.*

Theorem 2.17. *Every bipartite graph is move-equivalent to a reduced graph. A bipartite graph is reduced if and only if it has the minimal number of faces in its move-equivalence class. Any two reduced graphs in the same move-equivalence class are related by the equivalences (M1) and (M2).*

Proof. Omitted. □

A **bounded affine permutation**, or **bounded juggling pattern** is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying:

(1) $i \leq f(i) \leq i + n$
(2) $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$
(3) $\sum_{i=1}^{n}(f(i) - i) = kn$

If f is a bounded affine permutation, then for $i \in \mathbb{Z}/n\mathbb{Z}$, fs_i is the bounded affine permutation obtained by swapping $f(j)$ with $f(j+1)$ for all $j = i \mod n$. The length $\ell(f)$ of a bounded affine permutation is the cardinality of the set of inversions:

$$\{(i, j) \in [n] \times \mathbb{Z} \mid i < j \text{ and } f(i) > f(j)\}.$$

The bounded affine permutation given by $f(i) = i + k$ is the unique element (for fixed k and n) with length 0.

If G is reduced, then we define a bounded affine permutation f_G by insisting that $f_G(i) = \pi_G(i) \mod n$. Given the bounded condition, the only time there is ambiguity is if the trip that starts at i ends at i. In this case we have $f_G(i) = i$ if i is incident to a black vertex and $f_G(i) = i + n$ if i is incident to a white vertex.

2.6. **From matchings to flows.** See the Problem Set.

3. Positroids and TNN Grassmann cells

3.1. **Positroids.** The TNN Grassmannian $\mathrm{Gr}(k, n)_{\geq 0}$ is the subset of $\mathrm{Gr}(k, n)$ represented by $k \times n$ matrices X such that all Plücker coordinates (that is, maximal $k \times k$ minors) $\Delta_I(X)$ are nonnegative. We first note that the cyclic group acts on $\mathrm{Gr}(k, n)$ with generator acting by the map

$$(v_1, v_2, \ldots, v_n) \to \left(v_2, \ldots, v_n, (-1)^{k-1} v_1\right)$$

where v_i are columns of some $k \times n$ matrix representing X.

If $X \in \mathrm{Gr}(k, n)_{\geq 0}$ we define

$$\mathcal{M}_X = \left\{ I \in \binom{[n]}{k} \mid \Delta_I(X) > 0 \right\}$$

to be the **positroid** (positive matroid) of X. Given a positroid $\mathcal{M}$, we let the positroid cell $\Pi_{\mathcal{M}}$ be

$$\Pi_{\mathcal{M}} = \{X \in \mathrm{Gr}(k, n)_{\geq 0} \mid \mathcal{M}_X = \mathcal{M}\}.$$

3.2. **Grassmann necklaces.** We write $\leq_a$ for the ordering $a < a+1 < \cdots < n < 1 < \cdots < a-1$ on $[n]$. We have $I = \{i_1 < i_2 < \cdots < i_k\} \leq J = \{j_1 < j_2 < \cdots < j_k\}$ if $i_r \leq j_r$ for all r. We also have the cyclically rotated version $I \leq_a J$.

Let $\mathcal{S}_I = \{J \in \binom{[n]}{k} \mid I \leq J\}$ be the Schubert matroid with minimal element I. Let $\mathcal{S}_{I,a} = \{J \in \binom{[n]}{k} \mid I \leq_a J\}$. (Exercise: prove these are matroids.)

Given $X \in \mathrm{Gr}(k,n)$ we write $X \in \Omega_I$ if I is the lexicographically minimal subset such that $\Delta_I(X) \neq 0$.

Lemma 3.1. *If $X \in \Omega_I$ then $\mathcal{M} \subset \mathcal{S}_I$.*

A **Grassmann necklace** is a collection of k-element subsets $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ satisfying the following property: for each $a \in [n]$:

(1) $I_{a+1} = I_a$ if $a \notin I_a$
(2) $I_{a+1} = I_a - \{a\} \cup \{a'\}$ if $a \in I_a$.

Note that in (2) a' is allowed to be a. To each $\mathcal{M}$ we associate the collection $\mathcal{I}(\mathcal{M}) = (I_1, \ldots, I_n)$ such that I_a is the lexicographically minimal basis with respect to $\leq_a$. We also write $\mathcal{I}(X)$ for $\mathcal{I}(\mathcal{M}_X)$.

Proposition 3.2. *$\mathcal{I}(X)$ is a Grassmann necklace.*

To a Grassmann necklace $\mathcal{I}$ we associate a bounded affine permutation $f(\mathcal{I})$ by $f(a) = a'$, with the following conventions: if $a \notin I_a$ then $f(a) = a$, and if $a \in I_a \cap I_{a+1}$ then $f(a) = a + n$.

Proposition 3.3. *The map $\mathcal{I} \mapsto f(\mathcal{I})$ is a bijection between Grassmann necklaces and bounded affine permutations.*

Proposition 3.4. *The bounded affine permutation f_X is given by*

$$f_X(i) = \min\{j \geq i \mid v_i \in \mathrm{span}\{v_{i+1}, v_{i+2}, \ldots, v_j\}\} \tag{2}$$

where v_i are the columns of a representative of X, and we extend these columns periodically.

3.3. **Reduction of TNN Grassmann cells.** Let $X \in \mathrm{Gr}(k,n)_{\geq 0}$. Suppose f_X has a fixed point $f_X(i) = i$. Then by (2), the i-th column v_i of any representative of X must be the 0 vector. We have a projection map $p_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ removing the i-th coordinate.

Lemma 3.5. *The projection map induces a bijection between $\{X \in \mathrm{Gr}(k,n)_{\geq 0} \mid f_X(i) = i\}$ and $\mathrm{Gr}(k, n-1)_{\geq 0}$.*

Now suppose f_X satisfies $f_X(i) = i + n$. Then by (2), the i-th column v_i of any representative of X is not in the span of the other columns. Treating X as a k-dimensional subspace of $\mathbb{R}^n$, we have that $p_i(X)$ is a $(k-1)$-dimensional subspace of $\mathbb{R}^n$.

Lemma 3.6. *The projection map gives a bijection between $\{X \in \mathrm{Gr}(k,n)_{\geq 0} \mid f_X(i) = i + n\}$ and $\mathrm{Gr}(k-1, n-1)_{\geq 0}$.*

Proof. By cyclic rotation we assume that $i = 1$. By left multiplying by $g \in GL(k, \mathbb{R})$, we may assume that the first column is $(1, 0, \ldots, 0)^T$ and that the first row is $(1, 0, \ldots, 0)$. It is clear that removing the first row and column gives a $(k-1) \times (n-1)$ matrix, representing a point in $\mathrm{Gr}(k-1, n-1)_{\geq 0}$, and that this is a bijection. □

We now give a bridge (or Chevalley generator) reduction of TNN points in the Grassmannian. Let X be a TNN point of the Grassmannian. Suppose the bounded affine permutation f_X satisfies $i < i+1 \leq f(i) < f(i+1) \leq i+n$. Then we say that X has a bridge at i.

For $1 \leq i \leq n-1$, we define $x_i(a)$ to be the $n \times n$ matrix which differs from the identity matrix by the entry a in the $(i, i+1)$ entry. If $X \in \mathrm{Gr}(k, n)$ is represented by a $k \times n$ matrix, then $X \cdot x_i(a)$ is obtained from X by adding a times the i-th column to the $i+1$-st column. For $i = n$, we should think of $x_n(a)$ as the operation obtained from $x_1(a)$ by conjugating by the generator of the $\mathbb{Z}/n\mathbb{Z}$ action on $\mathrm{Gr}(k, n)$.

Proposition 3.7. *Suppose $X \in \mathrm{Gr}(k, n)_{\geq 0}$ has a bridge at i. Then $a = \Delta_{I_{i+1}}(X)/\Delta_{I_{i+1} \cup \{i\} - \{i+1\}}(X)$ is positive and well defined, and $X' = X \cdot x_i(-a) \in \mathrm{Gr}(k, n)_{\geq 0}$ has a positroid strictly smaller than $\mathcal{M}_X$. We also have $f_{X'} = f_X s_i$.*

Proof. Let v_i be the columns of a $k \times n$ matrix which represents X.

If $f(i) = i+1$, then by (2), the columns v_i and v_{i+1} are parallel, and since $f(i+1) \neq i+1$ both v_i and v_{i+1} are non-zero. In this case a is just the ratio v_{i+1}/v_i, and X' is what we get by changing the $(i+1)$-st column to 0. All the claims follow.

We now assume that $f(i) > i+1$. For simplicity of notation, assume $i = 1$. Let $f(i) = j$ and $f(i+1) = k$. Since $f(i) \notin \{i, i+n\}$, we have $i \in I_i$ and $i \notin I_{i+1}$. We also have $i+1 \in I_i \cap I_{i+1}$. We let $I_i = \{i, i+1\} \cup I$, $I_{i+1} = (i+1) \cup I \cup \{j\}$, and $I_{i+2} = I \cup \{j, k\}$ for some $I \subset [n] - \{i, i+1\}$. Note that if $k = n+i$, then $I_{i+2} = I \cup \{j, i\}$; this immediately gives $\Delta_{i \cup I \cup j} \neq 0$.

Suppose $k \neq n+i$. Then we have a Plücker relation

$$\Delta_{i \cup I \cup j} \Delta_{(i+1) \cup I \cup k} = \Delta_{i \cup I \cup k} \Delta_{(i+1) \cup I \cup j} + \Delta_{i \cup (i+1) \cup I} \Delta_{I \cup j \cup k}$$

where all subsets are ordered according to $\leq_i$. (The easiest way to see that the signs are correct is just to take $i = 1$.) Since the RHS is positive, $\Delta_{i \cup I \cup j} \neq 0$.

Now X' is obtained from X by adding $-a$ times v_i to v_{i+1}. So

$$\Delta_J(X') = \begin{cases} \Delta_J(X) - a\Delta_{J - \{i+1\} \cup \{i\}}(X) & \text{if } i+1 \in J \text{ and } i \notin J \\ \Delta_J(X) & \text{otherwise.} \end{cases} \tag{3}$$

The formulae above are the minors of this specific representative of X'; the Plücker coordinates of the actual point in the Grassmannian are only determined up to a scalar. By Lemma 3.8 below, we see that $X' \in \mathrm{Gr}(k, n)_{\geq 0}$, and that $J \in \mathcal{M}_{X'}$ only if $J \in \mathcal{M}_X$. However, $\Delta_{I_{i+1}}(X') = 0$, so $\mathcal{M}_{X'} \subsetneq \mathcal{M}_X$.

Finally, let v_i' be the columns for the matrix obtained from v_i by right multiplication by $x'(-a)$. Then $\mathrm{span}(v_i) = \mathrm{span}(v_i')$ and $\mathrm{span}(v_i, v_{i+1}) = \mathrm{span}(v_i', v_{i+1}')$, so $f_{X'}(r) = f_X(r)$ unless $r \in \{i, i+1\} \mod n$. But $f_{X'} \neq f_X$ since $\Delta_{I_{i+1}}(X') = 0$. Thus $f_{X'}$ must be obtained from f_X by swapping the values of $f(i)$ and $f(i+1)$. □

Lemma 3.8. *Let $X \in \mathrm{Gr}(k,n)_{\geq 0}$ be as in Proposition 3.7, with $f(i) > i+1$. For simplicitly of notation suppose $i = 1$. Write $I_2 = 2 \cup I \cup j$. Suppose $J \subset \{3,\ldots,n\}$ satisfies $1 \cup J \in \mathcal{M}_X$. Then $\Delta_{1\cup I\cup j}(X)\Delta_{2\cup J}(X) \geq \Delta_{1\cup J}(X)\Delta_{2\cup I\cup j}(X)$.*

Proof. Let $\mathcal{M}$ be the positroid of X. We let $I_1 = \{1,2\} \cup I$, $I_2 = 2 \cup I \cup \{j\}$, and $I_3 = I \cup \{j,k\}$, as in the proof of Proposition 3.7. We have already shown in the proof of Proposition 3.7 that $(1 \cup I \cup j) \in \mathcal{M}$.

We proceed by induction on the size of $r = |(I \cup j) \setminus J|$. The case $r = 0$ is tautological. So suppose $r \geq 1$. We may assume that $1 \cup J \in \mathcal{M}$ for otherwise the claim is trivial. Applying the exchange lemma to $1 \cup J$ the element $a = \max(J \setminus (I \cup j)) \in J$ and the other base $1 \cup I \cup j$, we obtain $L = J - \{a\} \cup \{b\}$ such that $1 \cup L \in \mathcal{M}$.

We claim that $b < a$. To see this, note that $I_1 \leq (1 \cup J)$, which implies that $a > I \setminus J$. So the only way that b could be greater than a is if $b = j$, and $a < j$. But by assumption we also have $I_3 = I \cup \{j,k\} \leq_3 (1 \cup J)$ with $k \geq_2 j$. This is impossible since both k and j are greater than a, but we have $J \setminus I \subset [3,a]$ – the only element of $(1 \cup J) \setminus I$ that is greater than j or k in $\leq_3$ order is 1. Thus $b < a$.

So by induction we have that $\Delta_{2\cup L}/\Delta_{1\cup L} \geq \Delta_{2\cup I}/\Delta_{1\cup I}$, where in particular we have $(1 \cup L), (2 \cup L) \in \mathcal{M}$. It suffices to show that $\Delta_{2\cup J}/\Delta_{1\cup J} \geq \Delta_{2\cup L}/\Delta_{1\cup L}$.

We apply the Plücker relation to $\Delta_{2\cup J}\Delta_{1\cup L}$, swapping L with $(k-1)$ of the indices in $2 \cup J$ to get

$$\Delta_{1\cup L}\Delta_{2\cup J} = \Delta_{1\cup J}\Delta_{2\cup L} + \Delta_{12j_1j_2\cdots\hat{a}\cdots j_{k-1}}\Delta_{\ell_1\ell_2\cdots a\cdots\ell_{k-1}}.$$

We note that $\ell_1 < \ell_2 < \cdots < a < \cdots < \ell_{k-1}$ is actually correctly ordered, since L is obtained from J by changing a to a smaller number. So all factors in the above expression are nonnegative. The claim follows. ☐

Remark 3.9. *In the proofs above I used the fact that each $X \in \mathrm{Gr}(k,n)$ is represented by a $k \times n$ matrix. However, this can be avoided, and all the proofs carried out mentioning Plücker coordinates only. This is advantageous if we only know the Plücker relations, and don't know that the relations are realized by the Grassmannian.*

3.4. Network realizability of $\mathrm{Gr}(k,n)_{\geq 0}$.

Theorem 3.10.

1. *Every $X \in \mathrm{Gr}(k,n)_{\geq 0}$ is representable by a network N.*
2. *The map $\mathcal{M} \mapsto f_{\mathcal{M}}$ is a bijection between positroids and bounded affine permutations. The map $\mathcal{M} \mapsto \mathcal{I}(\mathcal{M})$ is a bijection between positroids and Grassmann necklaces.*
3. *For each positroid cell $\Pi_{\mathcal{M}}$ there is a reduced bipartite graph G such that $M_G : (\mathcal{L}_G)_{>0} \to \Pi_G = \Pi_{\mathcal{M}}$ is bijective. The bounded affine permutation of G is equal to $f_{\mathcal{M}}$.*
4. *$\Pi_{\mathcal{M}} \simeq \mathbb{R}^d_{>0}$ has dimension equal to the length of $k(n-k) - \ell(f_{\mathcal{M}})$.*

Proof. We establish the first statement completely first. We proceed by induction on n, and then by induction on $|\mathcal{M}|$.

Suppose $n = 1$, then X is representable by a network N with a single boundary vertex joined to a single interior vertex, which can be either black or white. This represents the unique points in $\mathrm{Gr}(0,1)_{\geq 0}$ and $\mathrm{Gr}(1,1)_{\geq 0}$. This is the base case.

Now suppose $X \in \mathrm{Gr}(k,n)_{\geq 0}$. If $f_X(i) \in \{i, i+n\}$, then we can apply the reductions of Lemma 3.5 and Lemma 3.6 to get some X' which by induction is represented by a network N'. To obtain N from N' we insert a lollipop (with any edge weight, they are all gauge equivalent) at position i. Note that $f_{X'}$ is determined comletely by f_X.

Thus we may suppose that $f_X(i) \notin \{i, i+n\}$. But then we can find some i such that $f_X(i) < f_X(i+1)$ satisfying the conditions of Proposition 3.7. Let $X' \in \mathrm{Gr}(k,n)_{\geq 0}$ be the TNN point of Proposition 3.7. Then by induction on $\mathcal{M}$, we may assume that X' is represented by a network N'. Let N be the network obtained from N' by adding a bridge between i and $i+1$, white at i and black at $i+1$. Lemma 2.11 then says that N represents X.

Thus every $X \in \mathrm{Gr}(k,n)_{\geq 0}$ is representable by a network N. We note that the entire recursion depends only on f_X: we can choose the underlying graph G of N to depend on f_X only. Thus for each bounded affine permutation f, there is a graph $G(f)$ which parametrizes all of $\{X \in \mathrm{Gr}(k,n)_{\geq 0} \mid f_X = f\}$. But the matroid of $M(N)$ depends only on G (as long as all edge weights are positive), so we have a bijection between positroids and bounded affine permutations, and in turn Grassmann necklaces.

We note that adding a bridge adds one face and hence one parameter to $(\mathcal{L}_G)_{>0}$. Adding lollipops do not change the number of faces. So $(\mathcal{L}_{G(f)})_{>0} \simeq \mathbb{R}^d_{>0}$ where d is the number of bridges used in the entire recursion. Furthermore, the edge weights of the bridges determine the graph up to gauge equivalence, or, equivalently, these edge weights are coordinates on $(\mathcal{L}_{G(f)})_{>0}$. But the labels of the bridges are uniquely recovered $X = M(N)$ by the recursive algorithm above. So the map $M_G : (\mathcal{L}_G)_{>0} \to \Pi_{\mathcal{M}}$ is a bijection, where $G = G(f_{\mathcal{M}})$. By Theorem 2.17, G is reduced sinc $M_G : (\mathcal{L}_G)_{>0} \to \mathrm{Gr}(k,n)$ is injective (or the reduced statement can be proved directly).

Finally, we note that the dimension claim is true for $n = 1$, and we have $\ell(fs_i) = \ell(f) + 1$ when $f(i) < f(i+1)$. Now suppose we have X such that $f_X(i) = i$ and X' is obtained by the projection p_i. Then $\{(i,j) \mid i < j \text{ and } f_X(i) > f_X(j)\} = \emptyset$, but $|\{(j,i) \mid j < i \text{ and } f_X(j) > f_X(i)\}| = k$. So $\ell(f_X) = \ell(f_{X'}) + k$. A similar relation holds when $f_X(i) = i+n$. Thus the formula for the dimension of $\Pi_{\mathcal{M}}$ holds by induction. ☐

Proposition 3.11. *As N varies, all relations among the boundary measurements $\Delta_I(N)$ are generated by quadratic Plücker relations, or equivalently by the equality in Proposition 2.3.*

Proof. Let $\mathcal{M}$ be the uniform matroid. The top-dimensional positroid cell $\Pi_{\mathcal{M}}$ has the same dimension as the Grassmannian, which is irreducible. Thus $\Pi_{\mathcal{M}}$ is dense in $\mathrm{Gr}(k,n)$. But there is a graph G such that $(\mathcal{L}_G)_{>0} \simeq \Pi_{\mathcal{M}}$, and so any relations that holds for all N with underlying graph G must hold in $\mathrm{Gr}(k,n)$. The (homogenous) ideal of relations among Plücker coordinates is known to be generated by quadratic Plücker relations.

The last statement follows from Proposition 2.8: the set $\{\Delta_{\tau,T}\}$ forms a basis of the degree two part of the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$. ☐

3.5. **Plücker coordinates for the TNN Grassmannian are "log-concave".** Let $I = \{i_1 < i_2 < \ldots, i_k\}$, $J = \{j_1 < \cdots < j_k\} \in \binom{[n]}{k}$. Suppose the multiset $I \cup J$, when sorted, is equal to $\{a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_k \leq b_k\}$. Then we define $\mathrm{sort}_1(I,J) =$

$\{a_1, \ldots, a_k\}$ and $\mathrm{sort}_2(I, J) = \{b_1, \ldots, b_k\}$. Also for $I \cap J = \emptyset$ define $\min(I, J) = \{\min(i_1, j_1), \ldots, \min(i_k, j_k)\}$ and if $I \cap J \neq \emptyset$ define $\min(I, J) = (I \cap J) \cup \min(I \setminus J, J \setminus I)$. Similarly define $\max(I, J)$.

Proposition 3.12. *Let $X \in \mathrm{Gr}(k, n)_{\geq 0}$. Then*

$$\Delta_I(X)\Delta_J(X) \leq \Delta_{\min(I,J)}(X)\Delta_{\max(I,J)}(X) \leq \Delta_{\mathrm{sort}_1(I,J)}(X)\Delta_{\mathrm{sort}_2(I,J)}(X).$$

Proof. Follows from Proposition 2.3. □

A matroid $\mathcal{M}$ is **sort-closed** if $I, J \in \mathcal{M}$ implies $\mathrm{sort}_1(I, J), \mathrm{sort}_2(I, J) \in \mathcal{M}$.

Corollary 3.13. *Positroids are sort-closed.*

In fact the converse of Corollary 3.13 also holds.

3.6. Oh's theorem.

A consequence of Corollary 3.13 is Oh's theorem.

Theorem 3.14. *Positroids are intersections of cyclically rotated Schubert matroids: if $\mathcal{I}(\mathcal{M}) = (I_1, I_2, \ldots, I_n)$ then*

$$\mathcal{M} = \mathcal{S}_{I_1,1} \cap \mathcal{S}_{I_2,2} \cap \cdots \cap \mathcal{S}_{I_n,n}.$$

Theorem 3.14 follows from Corollary 3.13 via the theory of **alcoved polytopes** [LP].

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