Math 669 Winter 2024 Problem Sheet

Solutions must be typeset in LaTeX. Please acknowledge sources and other students you have worked with.

1. Prove that $\lambda \geq \mu$ in dominance order if and only if $\mu' \geq \lambda'$ in dominance order.

2. Prove that dominance order on partitions with at most m parts corresponds to containment of convex hulls of S_m orbits. For example, $(3,0,0)$ dominates $(2,1,0)$ because the convex hull of $(3,0,0),(0,3,0)$, and $(0,0,3)$ contains $(2,1,0),(1,2,0),\ldots$

3. Find and prove an analogue of growth diagrams that corresponds to the RSK correspondence. Namely, generalize the growth diagrams from class from standard Young tableaux to SSYT.

4. Let $P(t) = p_1 + p_2 t + p_3 t^2 + \cdots$. Prove that $P(t)H(t) = \frac{d}{dt}H(t)$, and hence obtain a recursion which determines p_i's and h_i's in terms of each other.

5. The sign $\mathrm{sign}(T)$ of a SYT T is the sign of the reading word of T, obtained by reading the rows of T from left to right, starting with the top row. Suppose that $w \mapsto (P,Q)$ under RSK. Show that $\mathrm{sign}(w) = (-1)^{v(\lambda)}\mathrm{sign}(P)\mathrm{sign}(Q)$, where $v(\lambda)$ is the maximum number of disjoint vertical dominoes that fit inside λ.

6. Let $p(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d$. Express $\omega(\prod_{i=1}^{\infty} p(x_i))$ as an infinite product. (Hint: first consider the case $p(x) = 1 + ax$.)

7. Let $f : \Lambda_{\mathbb{R}} \to \mathbb{R}$ be an $\mathbb{R}$-algebra morphism such that $f(s_\lambda) \geq 0$ for all λ. Prove that the generating function $a(t) = 1 + f(h_1)t + f(h_2)t^2 + \cdots$ is meromorphic. (Bonus: prove that all the zeroes of $a(t)$ are real and negative.)

8. Prove that

$$\prod_i 1/(1-qx_i) \prod_{i<j} 1/(1-x_i x_j) = \sum_\lambda q^{c(\lambda)} s_\lambda,$$

where $c(\lambda)$ is the number of parts of λ' that are odd.

9. Expand $\prod_{i=1}^{\infty}(1 + x_i + x_i^2)$ in terms of elementary symmetric functions.

10. Let $\delta_n = (n-1, n-2, \ldots, 1)$. Prove that

$$s_{\delta_n}(x_1, ..., x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j).$$

11. Let $n = ab$ and $w \in S_n$. Prove that if RSK sends w to (P,Q), and P,Q have shape the $a \times b$ rectangle, then all the insertion paths during RSK are vertical (bump straight down).

12. (This is a long problem, and can be counted as two.). We consider a discrete dynamical system, the states of which are infinite strings of 0-s and 1-s, with finitely many 1-s. The time evolution is obtained by the rule "move each 1 to the first position with a 0 to its right, starting with the leftmost 1, and moving each 1 exactly once". For example,

$t=0$	000110101011000000
$t=1$	000001010100111000
$t=2$	000000101010000111

I've highlighted in blue and red the trajectory of the leftmost two 1-s. These finite strings should be imagined within a sea of 0-s.

(a) In the above example, after $t = 1$, the dynamical system behaves rather predictably. Namely, there are a number of consecutive strings $11\cdots1$ of 1-s, called *solitons*, and a soliton of size k travels at speed k. In our example, there are three solitons of size 1 and one soliton of size 3.

Prove that this "soliton" behavior happens for large t, regardless of the initial configuration.

(b) Extract a permutation w from 10-sequence as follows. Change 1-s to (-s and 0-s to)-s, throwing away extra)-s so that a legal parenthesization is obtained. Now label the open parentheses $1, 2, \ldots$ from left to right, and then label the matching close parentheses by the corresponding number. Read the labels of the close parentheses from left to right. In our example, we would get

$$(()()()(()))\longrightarrow 122'33'44'566'5'1'\longrightarrow 234651.$$

where we used primes to indicate close parentheses.

Prove that if $w\mapsto(P,Q)$ under RSK, then the column lengths of $\mathrm{sh}(P)$ gives the lengths of solitons. In our example, the P tableau is

1	3	4	5
2			
6			

agreeing with the soliton lengths of $3,1,1,1$.

13. Work out and prove a "dual" version of RSK insertion that proves the dual Pieri rule bijectively:

$$e_k s_\lambda=\sum_\mu s_\mu$$

summed over partitions μ such that μ/λ is a vertical strip of size k.

14. Prove the dual Pieri rule by using the usual Pieri rule and induction, together with the identity

$$e_n-e_{n-1}h_1+e_{n-2}h_2-\cdots\pm h_n=0.$$

15. Prove the dual Jacobi-Trudi formula using lattice paths, similar to our proof of the Jacobi-Trudi formula in class.

16. Let $p(n)$ denote the number of partitions of n. Show that $\det(p(i-j+1))_{i,j=1}^n$ is equal to one of $0,1,-1$. (This is a variant of Euler's pentagonal number theorem. You may assume that result if it is helpful.)

17. Let

$$s_{(a|b)}:=h_{a+1}e_b-h_{a+2}e_{b-1}+\cdots+(-1)^b h_{a+b+1}.$$

(a) If $a,b\geq 0$ show that $s_{(a|b)}$ is equal to the Schur function for the shape $\lambda=(a+1,1^b)$.

(b) If a or b is negative, then $s_{(a|b)}=0$ unless $a+b=-1$, in which case $s_{(a|b)}=(-1)^b$.

(c) Let λ be a Young diagram and define the Frobenius notation $(a_1,\dots,a_d|b_1,\dots,b_d)$ of λ as follows. Suppose the main diagonal of λ consists of d boxes $(1,1),(2,2),\dots,(d,d)$. Let a_i be the number of boxes in the i-th row of λ to the right of (i,i). Let b_i be the number of boxes in the i-th column of λ below (i,i). For example, if $\lambda=(5441)$, then its Frobenius notation is $(4,2,1|3,1,0)$. Show that

$$s_\lambda=\det(s_{(a_i|b_i)})_{i,j=1}^d.$$

18. Let us consider the specialization $\psi_n:\Lambda\to\mathbb{R}$ given by

$$x_i\longmapsto\begin{cases}1/n & \text{for } i\in[1,n]\\ 0 & \text{for } i>n.\end{cases}$$

Show that

$$\lim_{n\to\infty}\psi_n(s_\lambda)=\prod_{(i,j)\in\lambda}\frac{1}{h_{i,j}(\lambda)}$$

where $h_{i,j}(\lambda)$ is the hook-length of the box (i,j) in λ.

19. Let $U:\Lambda\to\Lambda$ be the linear operator given by multiplication by s_1. Let $D:\Lambda\to\Lambda$ be the linear operator given by the adjoint, $s_1^\perp$. Show that $DU-UD=\mathrm{Id}$.

20. Recall that $H_{\lambda\mu}$ is the transition matrix from $\{h_\lambda\}$ to $\{m_\mu\}$. Show that $H_{\lambda\mu}$ is equal to the number of double cosets $G'gG''$ for the subgroups $G'=S_{\lambda_1}\times\cdots\times S_{\lambda_\ell}$ and $G''=S_{\mu_1}\times\cdots\times S_{\mu_\ell}$ in $G=S_n$, where $n=|\lambda|=|\mu|$.

21. Recall the Kronecker product $\chi^\lambda \otimes \chi^\mu = \sum_\nu g_{\lambda\mu\nu}\chi^\nu$ on irreducible characters of S_n. We have the corresponding Kronecker or internal product

$$s_\lambda * s_\mu = \sum_\nu g_{\lambda\mu\nu} s_\nu.$$

(a) Show that $g_{\lambda\mu\nu}$ is invariant under permuting the three indices λ, μ, ν.
(b) Show that $e_n * f = \omega(f)$, for $f \in \Lambda$ of degree n.
(c) Show that for $f, g, h \in \Lambda$, we have

$$\langle f(xy), g \otimes h\rangle = \langle f, g * h\rangle.$$

Here, $f(xy)$ is the symmetric function in the variables $x_i y_j$, viewed as an element of $\Lambda(x) \otimes \Lambda(y)$.

22. Prove the identity

$$\sum_\lambda s_\lambda * s_\lambda = \frac{1}{\prod_{i\geq 1}(1-p_i)}.$$

23. Let $S : \Lambda \to \Lambda$ denote the antipode of the Hopf algebra of symmetric functions. Show that $S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$.

24. For $f, g \in \Lambda$, show that the plethysm $h_n[fg]$ is given by

$$h_n[fg] = \sum_{\lambda \vdash n} s_\lambda[f] s_\lambda[g].$$

25. Prove that

$$h_r[h_2] = \sum_\mu s_\mu$$

summed over partitions $\mu \vdash 2r$ such that all parts of μ are even.

26. A partition λ is a *n-core* if no n-ribbon can be removed from it. That is, there does not exist $\mu \subset \lambda$ such that λ/μ is a ribbon of size n. Given an arbitrary λ, we obtain a n-core $\mathrm{core}_n(\lambda)$ by repeatedly removing ribbons of size n from λ until it is no longer possible to do so. Show that $\mathrm{core}_n(\lambda)$ is well-defined.

27. Let $n \geq 1$. Prove that the generating function of n-cores is given by

$$\sum_{\lambda\ n\text{-core}} t^{|\lambda|} = \prod_{i\geq 1} \frac{(1-t^{ni})^n}{1-t^i}.$$

28. Prove that

$$p_n[h_k] = \sum_\mu \pm s_\mu$$

where the summation is over partitions μ satisfying: (a) μ has at most n parts, (b) $|\mu| = kn$, and (c) $\mathrm{core}_n(\mu) = \emptyset$.

29. A *domino* is a 2-ribbon. The *head* of a domino is the NE box of the domino. A horizontal domino strip is a skew shape λ/μ such that we have a sequence $\lambda^{(0)} = \mu \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$ where for each i, the skew shape $\lambda^{(i)}/\lambda^{(i-1)}$ is a domino D_i, and furthermore the head of D_i is to the east of the head of D_{i-1}.

(a) Verify that if λ/μ is a horizontal domino strip then the decomposition into dominos is unique, and that every column contains at most 2 boxes.

(b) A (semistandard) domino tableau T of shape λ is a chain $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$ where each $\lambda^{(i)}/\lambda^{(i-1)}$ is a domino horizontal strip. The weight of T is the vector $(\mathrm{wt}_1(T), \mathrm{wt}_2(T), \ldots)$ where $\mathrm{wt}_i(T)$ is the number of dominos in $\lambda^{(i)}/\lambda^{(i-1)}$. The spin of $\mathrm{spin}(T)$ is the number of vertical dominos in T. We define the domino tableaux generating function

$$G_\lambda(x;q) = \sum_T x^{\mathrm{wt}(T)} q^{\mathrm{spin}(T)}$$

summed over all domino tableaux of shape λ. Show that $G_\lambda(x;q)$ is symmetric (in $x_1, x_2, \ldots$).

30. Prove that the domino generating function G_λ is Schur positive, that is, the coefficient of a Schur function s_ν is a polynomial in q with nonnegative integer coefficients.

31. Prove the following "skew Pieri rule":

$$h_k s_{\lambda/\mu} = \sum_{r=0}^{k} (-1)^r \sum_{\rho/\nu} s_{\rho/\nu}$$

where the second sum is over all skew shapes ρ/ν such that ρ/λ is a horizontal strip with $(k-r)$ boxes, and μ/ν is a vertical strip with r boxes.

32. In class we sketched a proof that Schur polynomials are the irreducible characters of $\mathrm{GL}(n)$. Prove one or more of the following statements that came up in class (don't use the theorem that Schur polynomials are exactly the irreducible characters!):

(1) For a finite-dimensional polynomial representation V of $\mathrm{GL}(n)$, the character $\mathrm{char}(V) \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ is a symmetric polynomial with *integer* coefficients.
(2) Every irreducible polynomial representation V of $\mathrm{GL}(n)$ appears as a direct summand of $(\mathbb{C}^n)^{\otimes m}$ for sufficiently large m.

33. Explain how to interpret the Cauchy identity, and the Pieri rule in terms of $\mathrm{GL}(n)$-representations.

34. The *adjoint* representation of $\mathrm{GL}(n)$ is the n^2-dimensional representation, with $g \in \mathrm{GL}(n)$ acting on the space of $n \times n$ matrices by $g.X = gXg^{-1}$. Compute the character of the adjoint representation, expressing it in terms of Schur polynomials.

35. Prove the following plethystic identity:

$$(\sum_{n \geq 0} h_n)[e_1 + e_2] = \sum_{\lambda} s_\lambda.$$

36. Prove the statement of Schur-Weyl duality: the associative subalgebras of $\mathrm{End}(V^{\otimes m})$ generated by the actions of $\mathrm{GL}(V)$ and S_m respectively, are the commutants of each other inside $\mathrm{End}(V^{\otimes m})$.

37. Let $V = \mathbb{C}^2$. Compute the dimension of the image of the group algebra $\mathbb{C}[S_m]$ inside $\mathrm{End}(V^{\otimes m})$.

38. Prove that the Plücker map $\mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$ is injective, and the image is closed.

39. Let $x_n = 1/n^2$ for $n = 1, 2, \ldots$. It is known that with this specialization, we have $E(-t^2) = sin(\pi t)/\pi t$, where as usual $E(t)$ is the generating function of elementary symmetric functions. Use this to find the value $\zeta(2r)$ of the Riemann zeta function at even positive integers.

40. A *reverse plane partition* is a filling F of a Young diagram λ with numbers $1, 2, 3, \ldots$ which is weakly increasing along both rows and columns. Define the weight of a reverse plane partition F by

$$\mathrm{wt}(F) = (\alpha_1, \alpha_2, \ldots), \qquad \alpha_i = \#\text{columns containing an } i.$$

Define $g_\lambda = \sum_F x^{\mathrm{wt}(F)}$, where the summation is over all reverse plane partitions of shape λ. Prove that g_λ is a symmetric function. (Bonus: prove that g_λ is Schur-positive.)

41. Let V be a polynomial representation of $\mathrm{GL}(n)$ with character a symmetric polynomial $f(x_1, \ldots, x_n)$. Embed the symmetric group S_n inside $\mathrm{GL}(n)$ as the subgroup of permutation matrices. Give a formula for the Frobenius characteristic of the representation V, viewed as a representation of this S_n. (Hint: diagonalize permutation matrices.)

42. Prove that $H^*(Gr(k,n))$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[e_1, e_2, ..., e_k, h_1, h_2, ...h_{n-k}]$ by the ideal generated by the coefficients of the polynomial

$$(1 + e_1 t + e_2 t^2 + ... + e_k t^k)(1 - h_1 t + h_2 t^2 - ... + (-1)^{n-k} h_{n-k} t^{n-k}) - 1.$$

43. Show that the Schubert variety X_λ can equivalently be defined by the conditions $\dim(V \cap F_{n-k+i-\lambda_i}) \geq i$, as i varies over the rows such that (i, λ_i) is a corner of the Young diagram λ. Show that none of these conditions can be omitted.

44. Define a map $\mathrm{Gr}(k,n) \to \mathrm{Gr}(n-k,n)$ by sending a subspace $V \subset \mathbb{C}^n$ to the kernel of the dual homomorphism $\mathbb{C}^n \cong (\mathbb{C}^n)^* \to V^*$. Show that this map is an isomorphism, and that it takes a Schubert variety X_λ to a Schubert variety $X_{\lambda'}$.

45. (A bit long, and can count as two problems.) Let V be the vector space with basis given by all Young diagrams. For $i \geq 1$, let $u_i : V \to V$ be the linear operator that takes a Young diagram to λ to the Young diagram μ obtained by adding a box to the i-th column. If adding a box to the i-th column does not give a Young diagram, then $u_i \cdot \lambda = 0$. Similarly, define d_i for $i \geq 1$ by removing a box in the i-th column.
(a) Show that u_i, d_j satisfy the relations

$$\begin{aligned} d_j u_i &= u_i d_j, && i \neq j \\ d_{i+1} u_{i+1} &= u_i d_i, \\ d_1 u_1 &= \mathrm{Id}. \end{aligned}$$

(b) Define operators A_i (resp. B_i) on V by adding (resp. removing) horizontal strips of size i in all possible ways. So for example $A_2 \cdot (1) = (21) + (3)$ but $B_2 \cdot (1) = 0$. Define $A(x) = \sum_{i=0}^{\infty} A_i x^i$ and similarly $B(x)$. Show that

$$A(x) = \cdots (\mathrm{Id} + xu_3)(\mathrm{Id} + xu_2)(\mathrm{Id} + xu_1), \qquad B(x) = (\mathrm{Id} + xd_1)(\mathrm{Id} + xd_2)(\mathrm{Id} + xd_3)\cdots$$

(c) Use (a) and (b) to show that $B(y)A(x) = A(x)B(y)(1-xy)^{-1}$.
(d) Deduce the Cauchy identity from (c).

46. Define symmetric functions $q_0 = 1, q_1, q_2, \ldots$ by the generating function identity

$$\sum_{i=0}^{\infty} q_i t^i = Q(t) = \prod_j \frac{1 + x_j t}{1 - x_j t}.$$

(a) Show that q_{2m} belongs to the ring over $\mathbb{Q}$ generated by $q_1, q_3, \ldots, q_{2m-1}$.
(b) Let $\Gamma_\mathbb{Q} \subset \Lambda_\mathbb{Q}$ be the subring of symmetric functions generated by q_i, over the rational numbers. Show that

$$\Gamma_\mathbb{Q} = \mathbb{Q}[p_1, p_3, p_5, \ldots].$$

47. Define $\Gamma \subset \Lambda$ as the subring of symmetric functions generated by q_i, over the integers. Define a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ to be *strict* if $\lambda_1 > \lambda_2 > \cdots$. Show that $\{q_\lambda = q_{\lambda_1} \cdots q_{\lambda_\ell} \mid \lambda \text{ strict}\}$ is a $\mathbb{Z}$-basis of Γ.

48. Show that the Schubert cell $\Omega_\mu(F_\bullet)$ is in the closure of the Schubert cell $\Omega_\lambda(F_\bullet)$ if and only if $\lambda \subseteq \mu$.