Infinite reduced words

Thomas Lam

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Coxeter groups

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A word $i_1 i_2 \cdots i_\ell$ is a reduced word if ℓ is minimal amongst expressions $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w.

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A word $i_1 i_2 \cdots i_\ell$ is a reduced word if ℓ is minimal amongst expressions $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w.
An infinite reduced word is a sequence $i_1 i_2 i_3 \cdots$ such that each initial subsequence $i_1 i_2 \cdots i_k$ is reduced.

Example (Symmetric group S_3)

S_3 is generated by involutions s_1, s_2 with the relation

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Example (Affine symmetric group $\tilde{S}_3$)

$\tilde{S}_3$ is generated by involutions s_0, s_1, s_2 with relations

$$s_1 s_2 s_1 = s_2 s_1 s_2 \qquad s_0 s_1 s_0 = s_1 s_0 s_1 \qquad s_2 s_0 s_2 = s_0 s_2 s_0$$

$012012012012\cdots$ is an infinite reduced word

Davis complex

Let W be a Coxeter group.

Davis Complex

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- Chambers/alcoves in X are the same as elements in W
- Hyperplanes in X are the same as reflections in W

A reflection in W is an element conjugate to one of the s_i.

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- Chambers/alcoves in X are the same as elements in W
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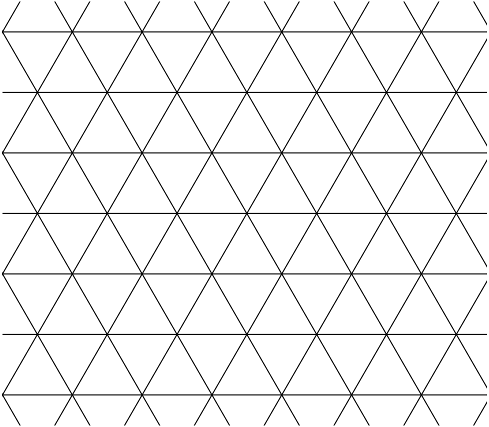
A reflection in W is an element conjugate to one of the s_i.

Compared to the Coxeter complex it has the following advantages:

1. The Davix complex is locally-finite. Only finitely many hyperplanes pass through each point.
2. The fundamental domain is compact.

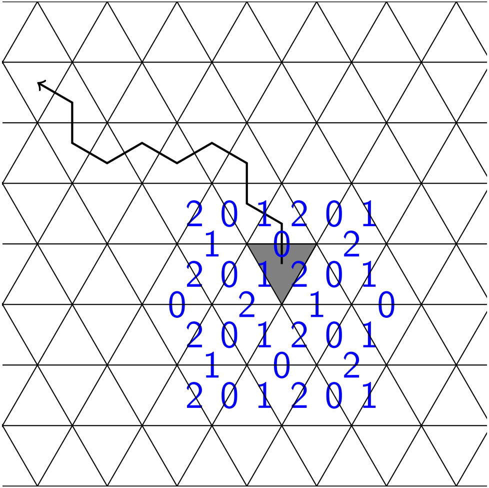
Because of these, and other nice metric properties ($\mathrm{CAT}(0)$), it is much more convenient to do geometric group theory on the Davis complex.

The $\tilde{A}_2$ Davis complex



The affine symmetric group $\tilde{S}_3$ acts simply-transitively on the alcoves of this arrangement.

Infinite reduced word = walk in Davix complex

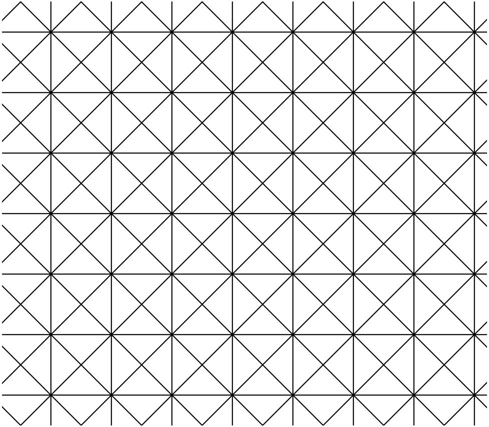


The above walk corresponds to the infinite reduced word $0120210201\cdots$.

REDUCED = no hyperplane crossed more than once

$\widetilde{A_1 \times A_1}$ Davix complex

$\tilde{B}_2$ Davis complex



A hyperbolic Coxeter group

Consider the Coxeter group

$$W = \langle s_1, s_2, s_3, s_4, s_5 \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle.$$

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Its Davis complex can be equipped with a piecewise hyperbolic metric:

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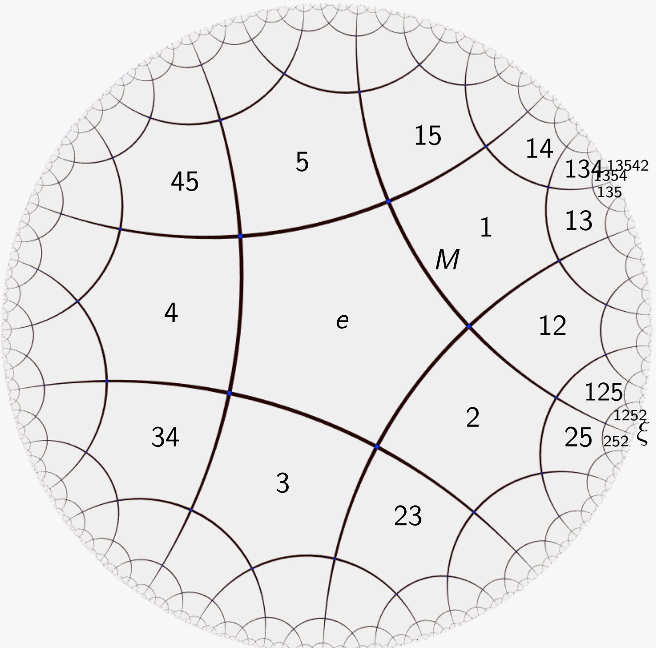
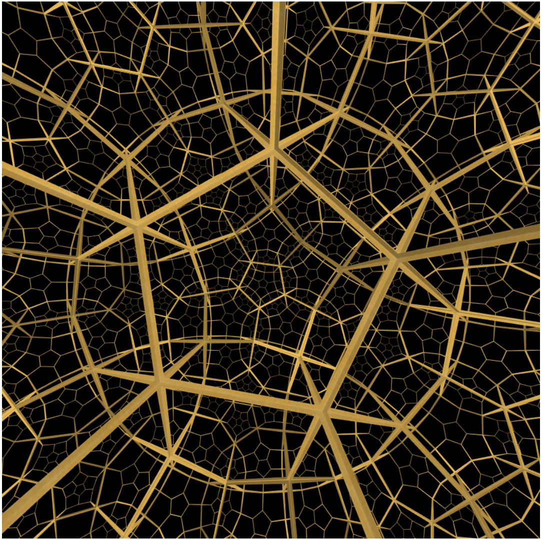


Another hyperbolic Coxeter group



Start with a regular dodecahedron in real hyperbolic 3-space, and start reflecting it assuming all dihedral angles are right angles.

Braid limits and braid equivalence

Assume from now on that W is an infinite Coxeter group.

Definition

There is a braid limit

$$\mathbf{i} = i_1 i_2 i_3 \cdots \longrightarrow \mathbf{j} = j_1 j_2 j_3 \cdots$$

if we can go from $\mathbf{i}$ to $\mathbf{j}$ by a (possibly infinite) sequence of braid moves for which every position eventually stabilizes.

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Example of braid limit

Take $n = 3$, and the affine symmetric group with simple generators s_0, s_1, s_2 satisfying

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Theorem (L.-Pylyavskyy)

Can always end up at an infinite power of a Coxeter element. For $n = 3$: $(012)^\infty$, $(120)^\infty$, $(201)^\infty$, $(210)^\infty$, $(102)^\infty$, $(021)^\infty$

Holds for any affine symmetric group.

Limit weak order

Definition

The limit weak order is the partial order on braid equivalence classes of infinite reduced words obtained from the preorder

$$\mathbf{j} \preceq \mathbf{i} \qquad \text{if} \qquad \mathbf{i} \to \mathbf{j}.$$

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Problem

Describe the braid equivalence classes and limit weak order.

Blocks

Lemma

i *and* **j** *are braid equivalent if and only if the corresponding walks cross exactly the same set of hyperplanes.*

Definition (L.-Pylyavskyy)

We say **i** and **j** are in the same block if the (infinite) set of hyperplanes they cross only differ by a finite set.

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***i** and **j** are braid equivalent if and only if the corresponding walks cross exactly the same set of hyperplanes.*

Definition (L.-Pylyavskyy)

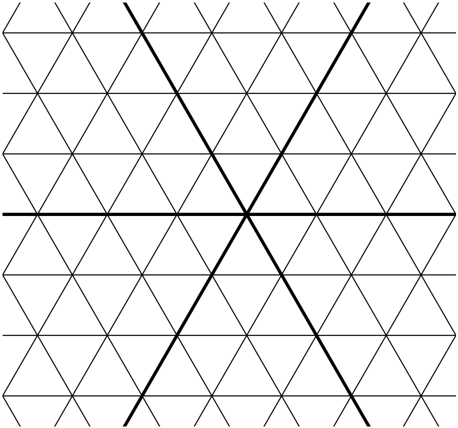
We say **i** and **j** are in the same block if the (infinite) set of hyperplanes they cross only differ by a finite set.

An affine Weyl group is a group generated by affine reflections acting cocompactly on a Euclidean space.

Theorem (L.-Pylyavskyy)

There is a bijection between blocks of an affine Weyl group W and faces of the braid arrangement of the finite Weyl group W_{fin} (with the origin omitted). The limit weak order is sent to inclusion order.

Braid arrangement for A_2



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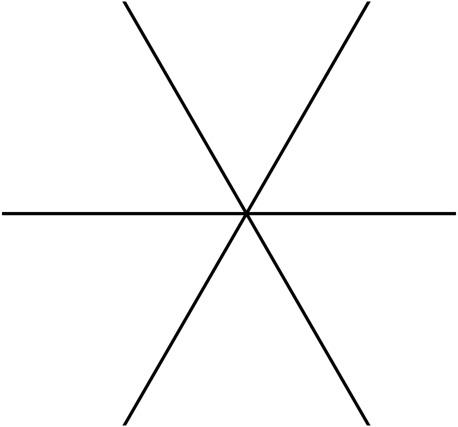

The braid arrangement is formed by the hyperplanes passing through the origin. Here there are six two-dimensional faces, and six one-dimensional faces. So $\tilde{S}_3$ has twelve blocks. The six one-dimensional faces correspond to $(012)^\infty$, $(120)^\infty$, $(201)^\infty$, $(210)^\infty$, $(102)^\infty$, $(021)^\infty$.

Beyond affine Coxeter groups

Visual boundary

The visual boundary of the Davis complex has underlying set given by equivalence classes of geodesic rays, where two geodesic rays are equivalent if they stay bounded distance apart.

The Tits boundary $\partial_T X$ is the visual boundary equipped with the Tits metric.

Tits boundary and limit weak order

For each infinite reduced word $\mathbf{i}$, define $\partial_T X(\mathbf{i}) \subset \partial_T X$ to consisting of geodesic rays which pass the same set of hyperplanes as $\mathbf{i}$.

Theorem (L.-Thomas)

1. *For each* $\mathbf{i}$ *and* $\mathbf{j}$ *we have* $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$ *or* $\partial_T X(\mathbf{i}) \cap \partial_T X(\mathbf{j}) = \emptyset$. *We have* $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$ *if and only if* $\mathbf{i}$ *and* $\mathbf{j}$ *are in the same block.*
2. *The* $\partial_T X(\mathbf{i})$ *form a partition of* $\partial_T X$.
3. *Each* $\partial_T X(\mathbf{i})$ *is a path-connected, totally geodesic subset of* $\partial_T X$.
4. *The closure of* $\partial_T X(\mathbf{i})$ *in* $\partial_T X$ *is the following union:*

$$\overline{\partial_T X(\mathbf{i})} = \bigcup_{\mathbf{j} \leq \mathbf{i}} \partial_T X(\mathbf{j})$$

where $\leq$ *denotes the limit weak order.*

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Question

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We can think of random infinite reduced words as infinite random walks in the Davis complex. These random walks will induce a measure on the partition $\partial_T X = \cup \partial_T X(\mathbf{i})$.

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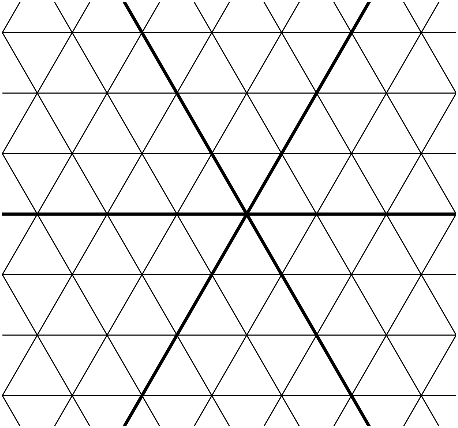
We will mostly focus on the case of affine Weyl groups.

Weyl chambers



The Weyl chambers are formed by the hyperplanes passing through the origin. Here there are six Weyl chambers, in bijection with the finite Weyl group S_3 (generated by reflections in these three hyperplanes).

Reduced random walk

Fix an affine Weyl group W.

The reduced random walk $X = (X_0, X_1, \ldots)$ is a sequence of alcoves in the Davis complex of W, where each step is chosen uniformly at random amongst choices which keep the walk reduced.

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Easy Facts:

1. These walks can never "get stuck".

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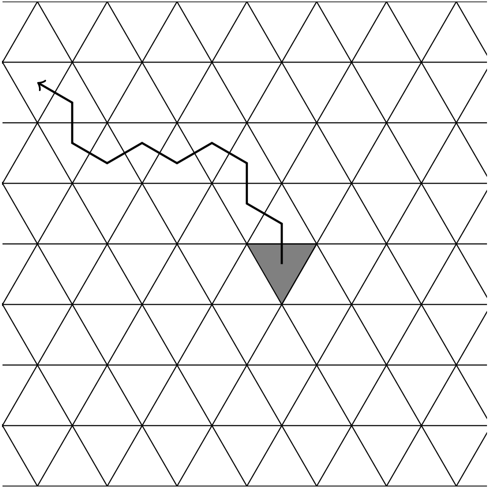
Easy Facts:

1. These walks can never "get stuck".
2. This process is a transient Markov chain.

Infinite reduced random walks



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Main Theorem 1

Fix an affine Weyl group W. Let $X = (X_0, X_1, \ldots)$ be the reduced random walk.

Theorem (L.)

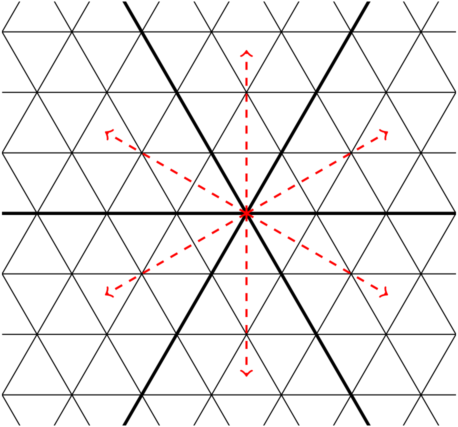
There exists a unit vector $\psi \in V$ such that almost surely

$$\lim_{k \to \infty} \nu(X_k) \in W \cdot \psi$$

where $\nu(X_k)$ denotes the unit vector pointing towards the center of the alcove X_k.

In other words, there is a finite collection $\{W \cdot \psi\}$ such that with probability one, the reduced random walk asymptotically approaches one of these directions.

Asymptotic directions



The asymptotic directions for $\tilde{S}_3$.

A Markov chain on W_{fin}

Define a Markov chain on the finite Weyl group W_{fin} with transitions of probability $1/r$ (with $r = \dim V + 1$) given by either

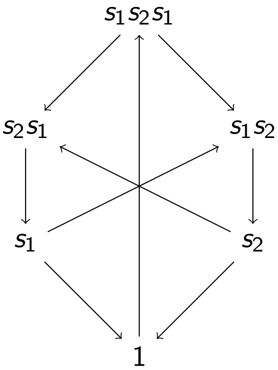
$$w \to s_i w \qquad \text{if } \ell(s_i w) < \ell(w)$$

or

$$w \to r_\theta w \qquad \text{if } \ell(r_\theta w) > \ell(w)$$

Here r_θ is the longest reflection in W_{fin}, and extra transitions from w to w are added to make this a Markov chain.

The Markov chain for S_3



All transitions have probability $1/3$. Add self-loops to make this a Markov chain.

Main Theorem 2

Theorem (L.)

The Markov chain on W_{fin} has a unique stationary distribution $\zeta : W_{\text{fin}} \to \mathbb{R}$. We have

$$\psi = \frac{1}{Z} \sum_{w \in W_{\text{fin}} : \ell(r_\theta w) > \ell(w)} \zeta(w) w^{-1}(\theta^\vee).$$

Since there are only finitely many Weyl chambers, the reducedness condition implies that every reduced walk will eventually stay in some Weyl chamber C_w. Write

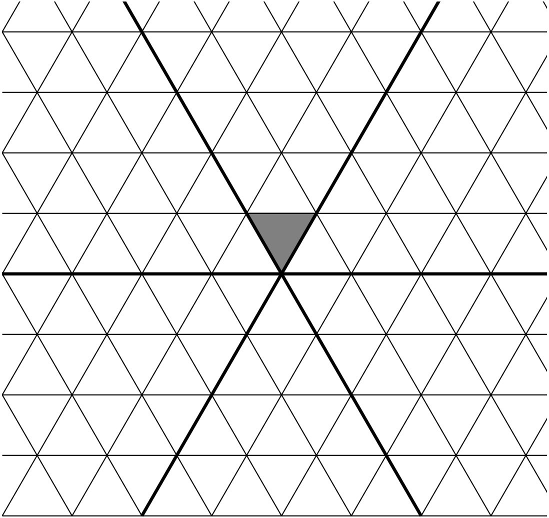
$$X \in C_w$$

for this event.

Question

What is $\mathrm{Prob}(X \in C_w)$?

Can you guess $\mathrm{Prob}(X \in C_w)$?



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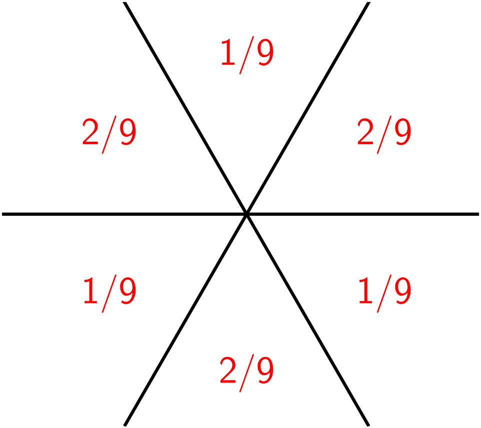

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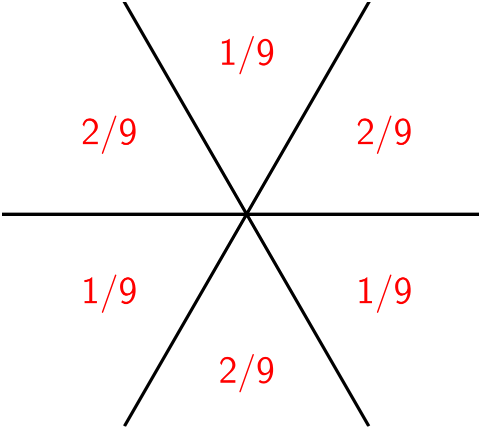

In four dimensions, one chamber is 96 times more likely than the least likely chamber.

Main Theorem 3

Theorem (L.)

$$\mathrm{Prob}(X \in C_w) = \zeta(w^{-1} w_0)$$

where $w_0 \in W_{\text{fin}}$ *is the longest element of* W_{fin}*.*

Type A case

Let $W = \tilde{S}_n$.
As observed by Ayyer and Linusson, the Markov chain on W_{fin} was previously studied by Ferrari and Martin under the name of multi-type TASEP on a circle. They gave a description of $\zeta(w)$ as counting certain multiline queues.

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Many more conjectures for a multivariate generalization of the Markov chain on S_n with L. Williams, suggesting very interesting enumeration!

n-cores

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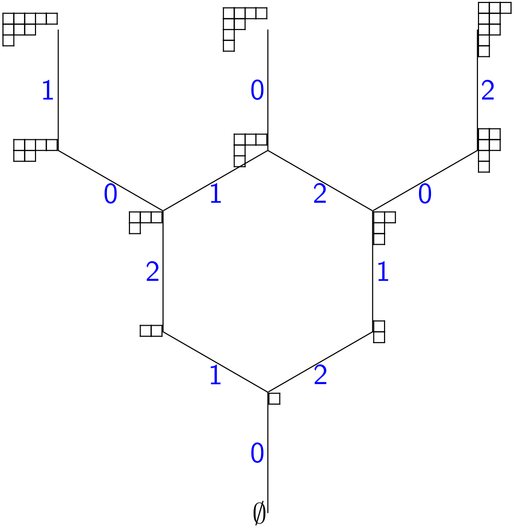
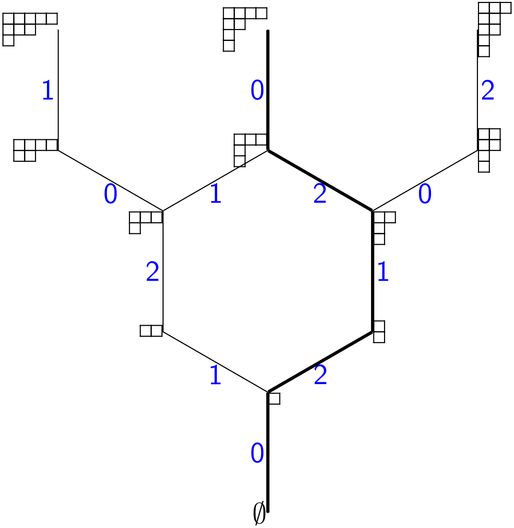


n-cores are a special class of partitions. Here we illustrate the bijection between 3-cores and Grassmannian elements of $\tilde{S}_3$.



The reduced word $02120\cdots$ gives the thickened line.

The limiting shape of a random n-core.

Corollary

There exists a piecewise-linear curve C_n such that most large random n-cores (grown by the "reduced" process) has a shape arbitrarily close to C_n.

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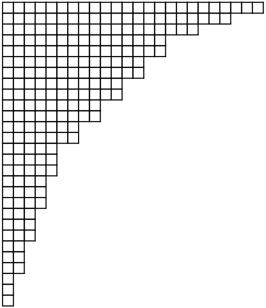

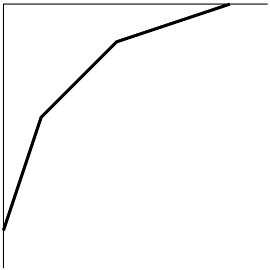

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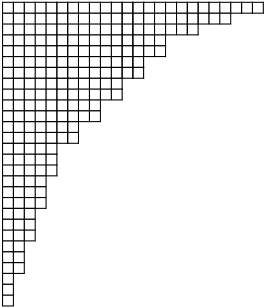
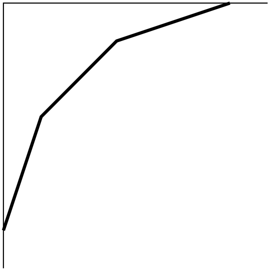




This might be compared with Kerov and Vershik's work on the shape of a random partition.

Taking $n \to \infty$

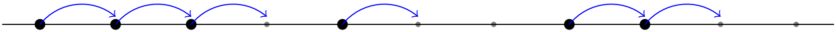
As $n \to \infty$, the piecewise-linear curve C_n, suitably scaled, approaches (one branch of) the continuous conic

$$\sqrt{x} + \sqrt{y} = 1$$

This curve has previously appeared as the limiting shape of another random process...

Continuous time TASEP

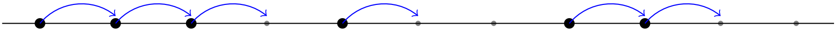
Continuous time TASEP on the integer lattice:



An independent random variable (waiting time) with exponential distribution is associated to each particle. The particle can jump only if the site immediately to the right is empty.

Continuous time TASEP

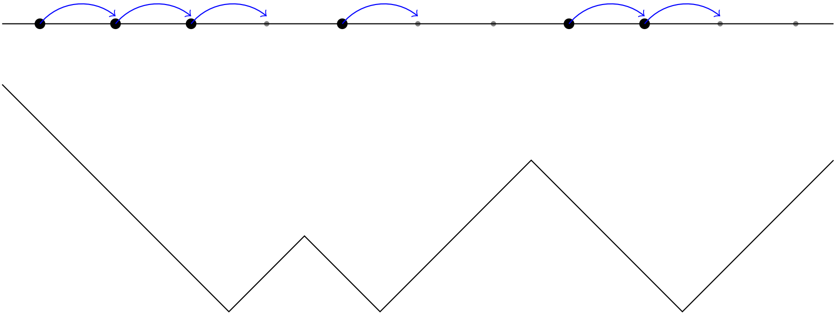
Continuous time TASEP on the integer lattice:



Initial configuration:

Continuous time TASEP

Continuous time TASEP on the integer lattice:



Each configuration is associated with a piece-wise linear curve, or Young diagram.

Continuous time TASEP

Continuous time TASEP on the integer lattice:

Johansson showed that the "limiting shape" of continuous time TASEP with exponential waiting time is exactly the same curve

$$\sqrt{x} + \sqrt{y} = 1$$

So for the affine symmetric group $W = \tilde{S}_n$, and conditioning our random walk to stay in the fundamental chamber, we obtain a periodic analogue of continuous time TASEP: particles separated by distance n are conditioned to jump simultaneously.