Mirror symmetry for flag varieties via Langlands reciprocity

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This is joint work with Nicolas Templier.

Quantum differential equations

M smooth compact Fano variety over $\mathbb{C}$
Examples: $\mathbb{P}^n, \mathrm{Gr}(k,n), G/P, \ldots$.
Fano index: largest integer m such that $-K_X = mD$ in $\mathrm{Pic}(M)$

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Quantum connection (Dubrovin) in anticanonical direction

Connection on the trivial $H^*(M)$-bundle over $\mathbb{C}_q^\times$:

$$\nabla = \nabla_M = d + D *_q \frac{dq}{q}$$

Here, $\log q$ is a coordinate on $\mathbb{C} \cdot [D] \subseteq H^2(X, \mathbb{C})$.
∇ is a connection on $\mathbb{C}_q^\times$ (regular at 0 and irregular at ∞).

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$M = \mathbb{P}^1$ with $\dim(H^*(\mathbb{P}^1)) = 2$

$$\left(q\frac{d}{dq} + \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} y_1(q) \\ y_2(q) \end{bmatrix} = 0$$

Landau-Ginzburg model

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{C}_q^\times & & \end{array}$$

$(X = \text{ smooth complex variety}, f, \pi)$

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Integral functions

$X_q := \pi^{-1}(q)$

$$\Psi(q) := \int_{\Gamma_q \subset X_q} e^{f(x)} \omega_q$$

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Landau-Ginzburg D-module

$\mathrm{Exp} := \mathbb{C}\langle x, \partial \rangle / (\partial x - x\partial - 1) = \mathbb{C}\langle x, \partial \rangle \cdot e^x$

$$\mathcal{C} = \mathcal{C}(X, f, \pi) := R\pi_! f^* \mathrm{Exp}.$$

Object in derived category of D-modules on $\mathbb{C}_q^{\times}$.

Givental's mirror conjecture

B-model — A-model

(X, f, π) LG-model — mirror — M Fano variety

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The integral functions $\Psi(q)$ are solutions to ∇_M.

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Stronger variant: D-module mirror conjecture

$\mathcal{C}(X, f, \pi)$ is a D-module and is isomorphic to ∇_M as D-modules on $\mathbb{C}_q^\times$.

Main Theorem

Theorem (L.-Templier)

Mirror conjecture holds for $M = G^\vee / P^\vee$ a minuscule flag variety, and Rietsch's LG-model (X, f, π).

Minuscule flag varieties:

- $\mathbb{P}^n$ (classical/Givental),
- $\mathrm{Gr}(k, n)$ (injection proved by Marsh-Rietsch),
- $\mathrm{OG}(n, 2n+1)$, $\mathrm{OG}(n, 2n)$,
- Q^{2n} (injection proved by Pech-Rietsch-Williams),
- Cayley plane,
- Freudenthal variety

Rietsch's LG-model

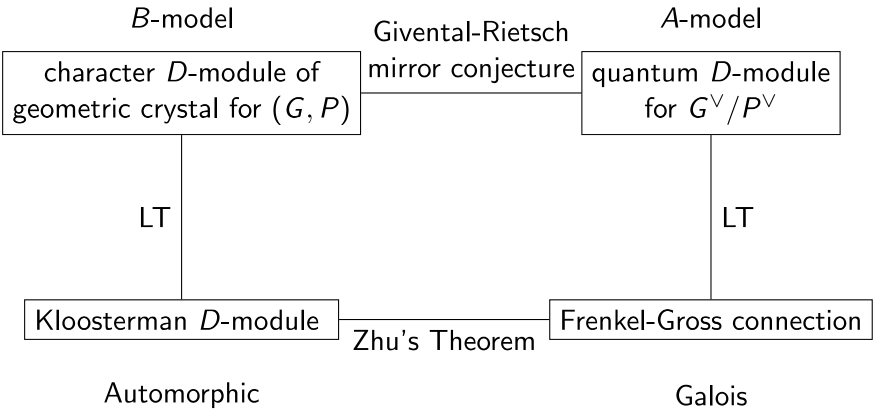
- (X, f, π) is a geometric crystal of Berenstein-Kazhdan,
- $\Psi(q)$ is a geometric character,
- $\mathcal{C}(X, f, \pi)$ is called the character D-module.

The fibers

$$X_q = \pi^{-1}(q) \simeq \mathring{G}/P \subset G/P$$

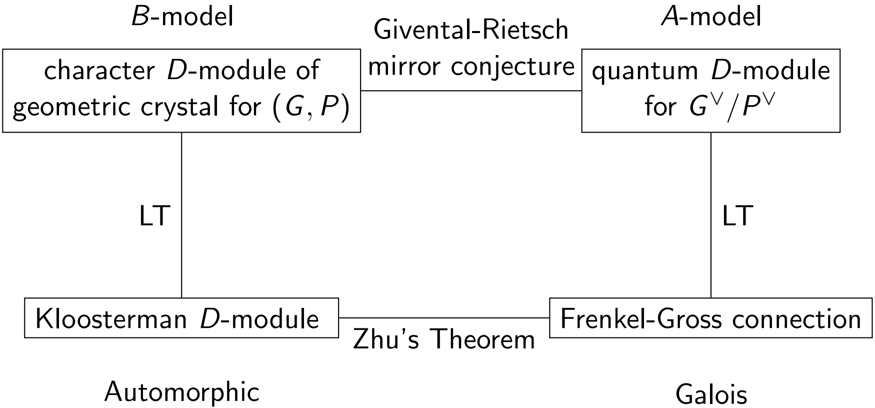
are isomorphic to a log Calabi-Yau subvariety called a projected Richardson variety (Lusztig, Rietsch, Goodearl-Yakimov, Knutson-L.-Speyer,..).

Proof idea



B-model
A-model
Givental-Rietsch mirror conjecture
character D-module of geometric crystal for (G, P)
quantum D-module for $G^\vee/P^\vee$
LT
LT
Kloosterman D-module
Frenkel-Gross connection
Zhu's Theorem
Automorphic
Galois

Proof idea



Zhu's theorem (based on Beilinson-Drinfeld) is an instance of the ramified geometric Langlands conjectures.

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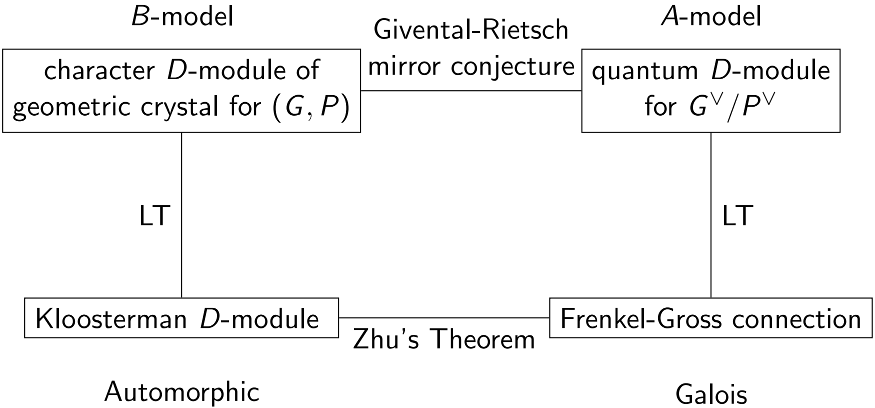

Zhu's theorem (based on Beilinson-Drinfeld) is an instance of the ramified geometric Langlands conjectures.
Right hand side is a calculation. Depends on some computations with canonical bases, and Mihalcea's quantum Chevalley formula.
(cf. Golyshev-Manivel in simply-laced cases)

Projective space case

$M = \mathbb{P}^{n-1} \qquad QH^*(\mathbb{P}^{n-1}) = \mathbb{C}[x,q]/(x^n - q).$

quantum D-module

$$q\frac{d}{dq} + \begin{bmatrix} 0 & 0 & \cdots & 0 & q \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = 0 \iff ((q\tfrac{d}{dq})^n - q)(\vec{y}(q)) = 0$$

(For $n = 1$: Bessel equation)

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LG-model

$$\begin{array}{ccc} (\mathbb{C}^\times)^n & \xrightarrow{f} & \mathbb{C} \\ \downarrow \pi & & \\ \mathbb{C}_q^\times & & \end{array} \qquad \begin{array}{ccc} (x_1, x_2, \ldots, x_n) & \longmapsto & x_1 + x_2 + \cdots + x_n \\ \downarrow & & \\ x_1 x_2 \cdots x_n & & \end{array}$$

Kloosterman sums

Base change to $\mathbb{F}_q$

$$\begin{array}{ccc} (\mathbb{F}_q^\times)^n & \xrightarrow{f} & \mathbb{F}_q \\ \downarrow \pi & & \\ \mathbb{F}_q^\times & & \end{array} \qquad \begin{array}{ccc} (x_1, x_2, \ldots, x_n) & \longmapsto & x_1 + x_2 + \cdots + x_n \\ \big\downarrow & & \\ x_1 x_2 \cdots x_n & & \end{array}$$

Kloosterman sums are analogues of the $\Psi(q)$

For $a \in \mathbb{F}_q^\times$, define

$$\mathrm{Kl}_n(a) := (-1)^{n-1} \sum_{x_1 x_2 \cdots x_n = a} \exp\left(\frac{2\pi i}{p} \mathrm{Tr} f(x)\right) \in \mathbb{C}$$

Here, $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$.

Weil-Deligne bound: $|Kl_n(a)| \leq nq^{(n-1)/2}$.

Kloosterman sheaves

Deligne (1970s): defined Kloosterman sheaf

$$\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell} := R\pi_! f^* \mathrm{AS}_\chi$$

where AS_χ is an Artin-Schreier sheaf. For suitable χ and $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$,

$$\mathrm{Kl}_n(a) = \iota \mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell})_a)$$

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$$\mathrm{Kl}_n(a) = \iota \mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell})_a)$$

Deligne: $\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell}$ is

- concentrated in degree 0 and is a local system
- tamely ramified at 0, maximal unipotent monodromy
- totally wildly ramified at ∞, Swan conductor equal to 1
- pure of weight $n-1$.

Katz: showed that $\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell}$ is rigid: determined by local monodromies
Gross ($\sim$2010): $F = \mathbb{F}_q(t)$, automorphic representation for $G(\mathbb{A}_F)$ for all semisimple G. For $G = GL_n$, the local representations matched the monodromies calculated by Deligne.

Heinloth-Ngo-Yun's automorphic sheaf

HNY's automorphic sheaf (a geometric version of Gross's automorphic representation)

$$\mathcal{A}_G \text{ on moduli stack of } \mathcal{G}\text{-bundles } \mathrm{Bun}_{\mathcal{G}} \text{ on } \mathbb{P}^1$$

The curve here is $\mathbb{P}^1_q \supset \mathbb{G}_m$.

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The curve here is $\mathbb{P}^1_q \supset \mathbb{G}_m$.
Here, $\mathcal{G}$ is a non-constant group scheme over $\mathbb{P}^1$, which is isomorphic to $G \times \mathbb{G}_m$ over $\mathbb{G}_m$. Behavior at 0 and ∞ encode information about the ramification.

Definition of Kloosterman D-module

$$\mathrm{Hecke} := \{(\mathcal{E}_1, \mathcal{E}_2, x \in \mathbb{G}_m, \phi : \mathcal{E}_1|_{\mathbb{P}^1 - x} \simeq \mathcal{E}_2|_{\mathbb{P}^1 - x})\}.$$

Hecke correspondence

$$\begin{array}{ccc} & \mathrm{Hecke} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathrm{Bun}_{\mathcal{G}} & & \mathrm{Bun}_{\mathcal{G}} \times \mathbb{G}_m \end{array}$$

Theorem (Heinloth-Ngo-Yun)

Heuristic version (actual version uses IC-sheaves):

$$Rp_{2,!}p_1^*\mathcal{A}_G \cong \mathcal{A}_G \boxtimes \mathrm{Kl}_{G^\vee}$$

where $\mathrm{Kl}_{G^\vee}$ *is the* $G^\vee$*-Kloosterman sheaf.*

Work over $\mathbb{C}$ with D-modules to define Kloosterman D-modules.
For $G^\vee = GL(n)$, recover Deligne's Kloosterman sheaf.

LG-models appear inside Hecke correspondence

Our idea: a piece of the Hecke correspondence

$$\begin{array}{ccc} & \mathring{\mathrm{Hecke}} & \\ \swarrow & & \searrow \\ \mathbb{G}_a^r \subset \mathrm{Bun}_{\mathcal{G}} & & * \times \mathbb{G}_m \subset \mathrm{Bun}_{\mathcal{G}} \times \mathbb{G}_m \end{array}$$

becomes isomorphic to $\begin{array}{ccc} & X & \\ {}^{f}\swarrow & & \searrow^{\pi} \\ \mathbb{C} & & \mathbb{C}_q^\times \end{array}$ after

- basechanging to $\mathbb{C}$
- composing with the sum map $\mathbb{G}_a^r \to \mathbb{G}_a$
- intersecting with a substack $\mathrm{Hecke}_\lambda \subset \mathrm{Hecke}$, whose fibers are finite-type $\mathrm{Gr}_\lambda \subset \mathrm{Gr}_G$.

Directions

- Other $G^\vee/P^\vee$?
- Hodge numbers of CY hypersurfaces $H \subset G^\vee/P^\vee$ vs. exponential hodge numbers of (X, f).
- Relation to Langlands functoriality: the quantum connection for $G^\vee/P^\vee$ is naturally a $G^\vee$ D-module, even though it is defined as a $\mathrm{GL}(H^*(G^\vee/P^\vee))$ D-module.
- For M arbitrary Fano, the quantum connection is a $\mathrm{GL}(H^*(M))$ θ-connection (Yun, Chen) built from picking a vector $X \in \mathfrak{g}_1$ in a Vinberg θ-group $\mathfrak{g} = \oplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$. It is irregular with slope $1/m$, where m is the Fano index.