Electroids and Positroids

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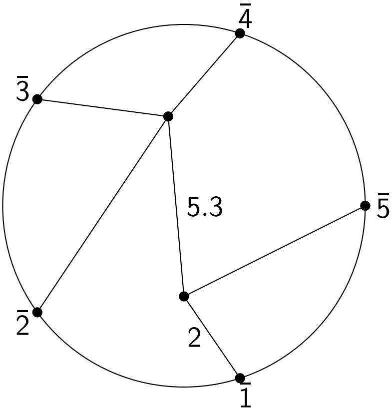
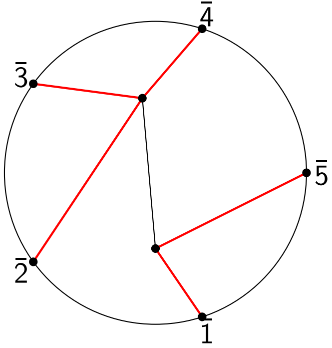
Circular planar electrical networks

We consider planar weighted graphs Γ embedded into the disk, with distinguished boundary vertices $\bar{1}, \bar{2}, \ldots, \bar{n}$ on the the boundary of the disk.



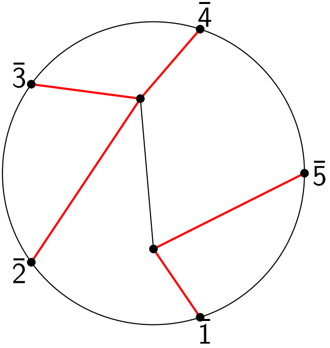
Kenyon-Wilson groves

A grove F in Γ is a subforest such that every interior vertex is connected to some boundary vertex.



Boundary partitions

The boundary partition $\sigma(F)$ of a grove F is the non-crossing partition whose parts are boundary vertices belonging to the same component of F.



$$\sigma(F) = \{\bar{2}, \bar{3}, \bar{4} | \bar{1}, \bar{5}\}$$

Planarity $\implies$ non-crossing.

Grove measurements

Let $\mathcal{NC}_n$ denote the set of non-crossing partitions on $\{\bar{1}, \ldots, \bar{n}\}$. For $\sigma \in \mathcal{NC}_n$, define the grove measurement

$$L_\sigma(\Gamma) = \sum_{\sigma(F)=\sigma} \mathrm{wt}(F)$$

where the weight of a grove F is the product of weights of edges belonging to F.

$$\Gamma \longmapsto \mathcal{L}(\Gamma) = (L_\sigma(\Gamma))_{\sigma \in \mathcal{NC}_n} \in \mathbb{P}^{\mathcal{NC}_n}.$$

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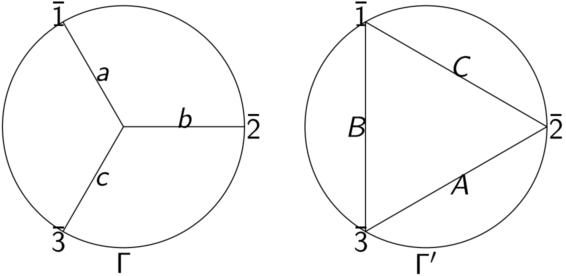
Two graphs are electrically equivalent if you cannot distinguish them by electrical experiments made at the boundary.

Theorem (Kirchhoff, Kenyon-Wilson)

Γ and Γ' are electrically equivalent if and only if $\mathcal{L}_\sigma(\Gamma) = \mathcal{L}_\sigma(\Gamma')$.

$Y - \Delta$-transformation



$$L_{\bar{1}|\bar{2}|\bar{3}|} = a + b + c, \qquad L_{\bar{1}\bar{2}|\bar{3}} = ab, \qquad L_{\bar{1}|\bar{2}\bar{3}} = bc,$$
$$L_{\bar{1}\bar{3}|\bar{2}} = ac, \qquad L_{\bar{1}\bar{2}\bar{3}} = abc$$

and

$$L'_{\bar{1}|\bar{2}|\bar{3}|} = 1, \qquad L'_{\bar{1}\bar{2}|\bar{3}} = C, \qquad L'_{\bar{1}|\bar{2}\bar{3}} = A,$$
$$L'_{\bar{1}\bar{3}|\bar{2}} = B, \qquad L'_{\bar{1}\bar{2}\bar{3}} = AB + BC + AC.$$

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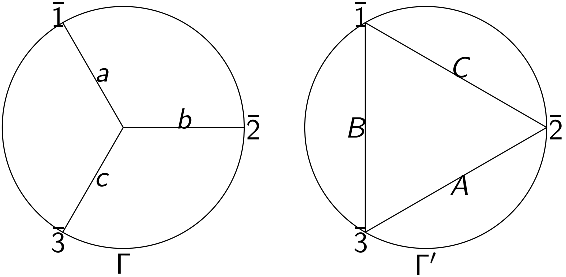

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$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c},$$

Compactification

Let us use nonnegative edge weights. The image of the map $\Gamma \mapsto \mathcal{L}(\Gamma)$ is not compact. We let

$$E_n \subset \mathbb{P}^{\mathcal{NC}_n}$$

denote the closure of the image, called the compactified space of circular planar electrical networks.

Roughly speaking, a point $\mathcal{L} \in E_n$ is represented by an electrical network where some of the boundary points have been glued together, in a planar way. (This is a good compactification. e.g. it's quite different from the one-point compactification.)

Electroids

The electroid $\mathcal{E}(\Gamma)$ of $\Gamma \in E_n$ is the set

$$\mathcal{E}(\Gamma) = \{\sigma \mid L_\sigma(\Gamma) \neq 0\} \subset \mathcal{NC}_n.$$

These are non-crossing partitions for which there exist groves inducing such a partition. (We think of this set as something like a matroid.)

Question

What are all possible electroids?

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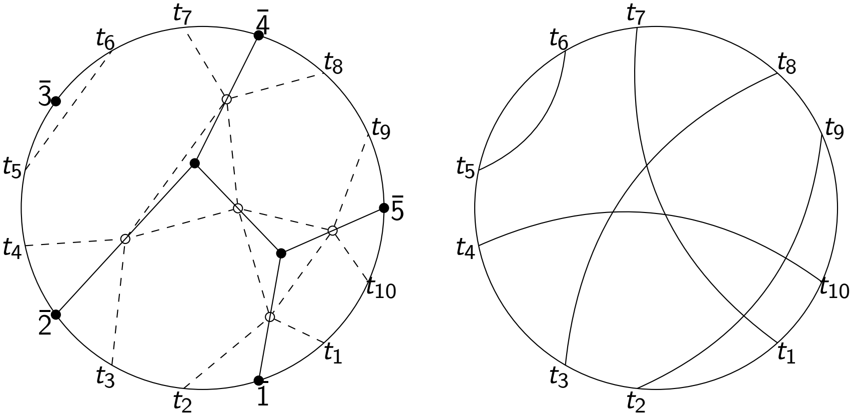
Question

What are all possible electroids?

We have the electroid stratification

$$E_n = \bigsqcup_{\mathcal{E}} E_{\mathcal{E}}.$$

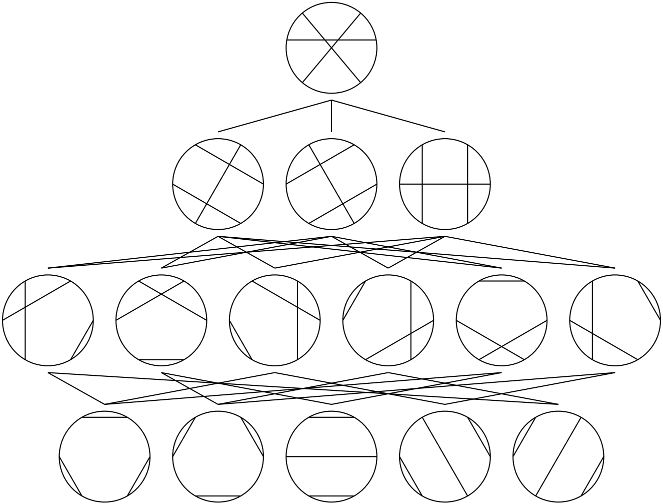
Medial graph



(Critical) Planar graph $\longrightarrow$ (Reduced) Medial graph $\longrightarrow$ Matching on $[2n]$

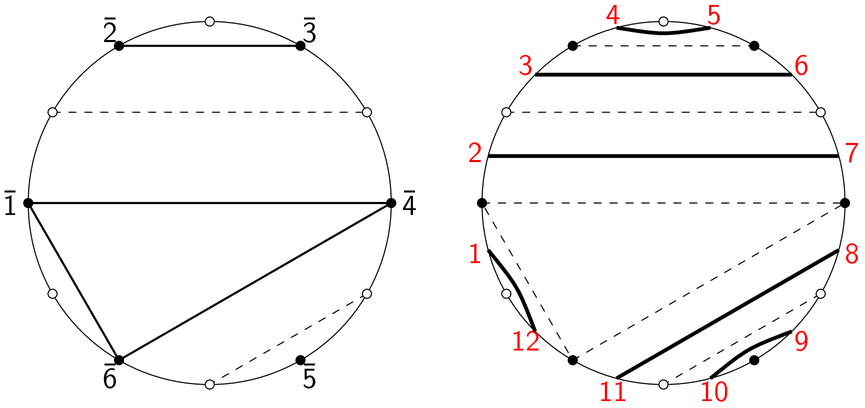
Uncrossing poset for matchings

The set P_n of matchings is a graded poset with rank function $c(\tau)$ = number of crossings. (Studied by Alman-Lian-Tran, by Kenyon, by Huang-Wen-Zie, by Kim-Lee...)



Noncrossing partitions to noncrossing matchings

For $\sigma \in \mathcal{NC}_n$, we have a natural $\tau(\sigma) \in P_n$ which is a noncrossing matching.



$$\sigma = (\bar{1}, \bar{4}, \bar{6} | \bar{2}, \bar{3} | \bar{5})$$

$$\tau(\sigma) = \{(1,12), (2,7), (3,6), (4,5), (8,11), (9,10)\}$$

Matchings classify electroid strata

Theorem (L.)

There is a bijection $\tau \leftrightarrow \mathcal{E}(\tau)$ *between matchings and electroids, given by*

$$\mathcal{E}(\tau) = \{\sigma | \tau(\sigma) \leq \tau\}$$

so that we have

$$E_n = \bigsqcup_{\tau \in P_n} E_\tau$$

where

$$E_\tau \simeq \mathbb{R}_{>0}^{c(\tau)}$$

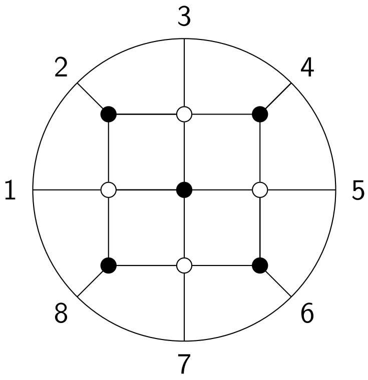
and

$$\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}.$$

This result depends on a large theory developed by Curtis-Ingerman-Morrow, and Colin de Verdière, Gitler, and Vertigan.

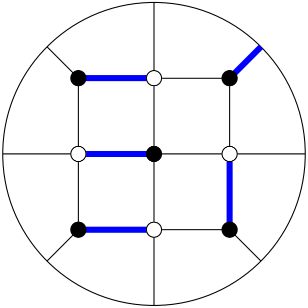
Planar bipartite graphs

Assumption: boundary vertices of N are always 1-valent.



Dimer configurations in planar bipartite graphs

Rule: Π must use all interior vertices; boundary vertices may or may not be used.



Boundary subset $I(\Pi)$ = black boundary vertices used union white boundary vertices not used.

Boundary measurements

$|I(\Pi)| = k(N)$ for some $k(N)$ that depends only on the planar bipartite graph N.

Define the boundary measurement

$$\Delta_I(N) = \sum_{I(\Pi)=I} \mathrm{wt}(\Pi).$$

The map

$$N \longmapsto M(N) = (\Delta_I(N))_{I \in \binom{[n]}{k}} \in \mathbb{P}^{\binom{[n]}{k}}$$

is a version of the dimer partition function.

TNN Grassmannian

Theorem (Kuo, Postnikov)

1. *The point* $M(N)$ *lies in the Grassmannian* $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{[n]}{k}}$.
2. *The image is the totally nonnegative Grassmannian* $\mathrm{Gr}(k,n)_{\geq 0}$ *consisting of points represented by nonnegative real coordinates.*

Analogies!

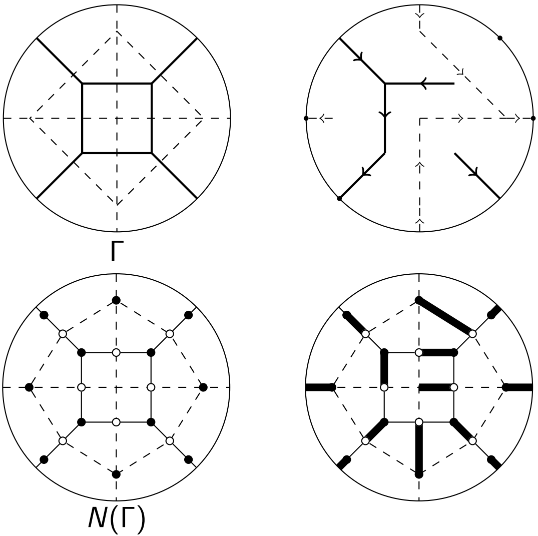
Planar bipartite graph N	Electrical network Γ
Dimer configurations in N	Groves in Γ
Plücker space $\mathbb{P}^{\binom{[n]}{k}}$	Non-crossing partition space $\mathbb{P}^{\mathcal{NC}_n}$
Grassmannian $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{[n]}{k}}$	Zariski closure of $E_n \subset \mathbb{P}^{\mathcal{NC}_n}$
Alternating strand diagram	Medial graph
Bounded affine permutations f	Medial pairings τ
Bruhat order	"Uncrossing" partial order
Subsets $I \in \binom{[n]}{k}$	Non-crossing partitions $\sigma \in \mathcal{NC}_n$
Positroids $\mathcal{M} \subset \binom{[n]}{k}$	Electroids $\mathcal{E} \subset \mathcal{NC}_n$
Grassmann necklaces	Partition necklaces
$GL(n)$-action	Electrical Lie group action
cluster algebra	Laurent phenomenon algebrfa

[Oh, Knutson-L.-Speyer, Thurston, Goncharov-Kenyon, Talaska, Postnikov-Speyer-Williams, ...]

[Curtis-Ingerman-Morrow, Colin de Verdière-Gitler-Vertigan, Kenyon-Wilson, L.-Pylyavskyy, ...]

Kenyon-Propp-Wilson's generalized Temperley trick

$$\Gamma \longmapsto N(\Gamma)$$



Everything is compatible

Theorem (L.)

The map $\Gamma \mapsto N(\Gamma)$ *induces an injection*

$$\iota : E_n \to \mathrm{Gr}(n-1, 2n)_{\geq 0}$$

given by

$$\Delta_I(N(\Gamma)) = \sum_{\sigma} a_{I\sigma} L_{\sigma(\Gamma)}$$

compatible with all the analogies.

The matrix $(a_{I\sigma})$ is a 0-1 matrix indexed by $\binom{[n]}{k} \times \mathcal{NC}_n$ which seems to be very interesting. It captures the algebraic structure of the generalized Temperley trick.

Posets

The closure partial order for planar bipartite case is given by a subposet

$$\mathrm{Bound}(k, n) \subset \tilde{S}_n$$

of the affine symmetric group equipped with Bruhat order.

Theorem (L.)

We have a map

$$\iota : P_n \hookrightarrow \mathrm{Bound}(n-1, 2n)$$

expressing P_n as an induced subposet.

Electroids and positroids

$$E_n = \bigsqcup_{\mathcal{E} \text{ electroid}} E_{\mathcal{E}} \qquad \text{and} \qquad \mathrm{Gr}(k,n)_{\geq 0} \bigsqcup_{\mathcal{M} \text{ positroid}} \mathring{\Pi}_{\mathcal{M}}.$$

Here a positroid $\mathcal{M}$ is the matroid of a point in $\mathrm{Gr}(k,n)_{\geq 0}$.

Theorem (L.)

The map $\iota : P_n \hookrightarrow \mathrm{Bound}(n-1, 2n)$ induces a map $\mathcal{E} \mapsto \mathcal{M}(\mathcal{E})$ such that

$$\iota(E_{\mathcal{E}}) = \iota(E_n) \cap \mathring{\Pi}_{\mathcal{M}(\mathcal{E})}.$$

The end

Thanks!