Lecture 3: Total positivity and combinatorial topology

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Thanks to Steven Karp for some slides!

CW complexes

- A *CW complex* is a Hausdorff topological space X together with a finite partition $X = \bigsqcup_{\alpha \in P} X_\alpha$ of *cells*, such that for each α,

  1. there is a continuous attaching map $f_\alpha : \overline{B}^d \to X$ defined on a closed ball, sending the open ball B^d homeomorphically onto X_α, and
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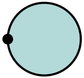
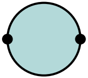
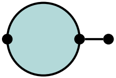
- e.g.  non-regular CW complex  regular CW complex  regular CW complex

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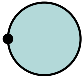
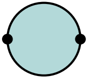
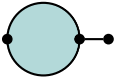
- e.g.  non-regular CW complex  regular CW complex  regular CW complex
- Any polytope Q is a regular CW complex.

CW posets

The *face poset* of a regular CW complex is the poset $(\hat{P} = P \sqcup \hat{0}, \preceq)$ where

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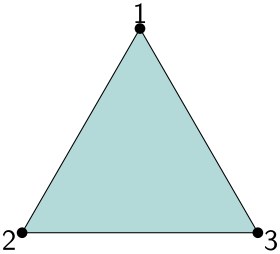
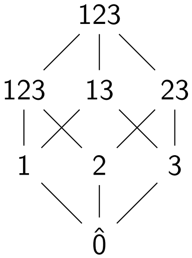
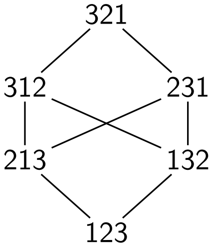
- Björner characterized the face posets of regular CW complexes, called *CW posets*.





Fomin–Shapiro conjecture



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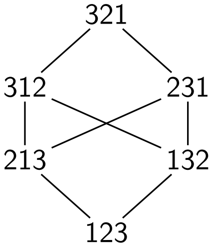

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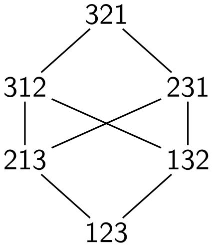

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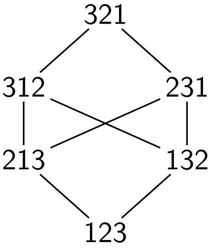

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$$Y_{\geq 0} := \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : \begin{matrix} x + z = 1, \\ \text{all minors} \geq 0 \end{matrix} \right\}$$

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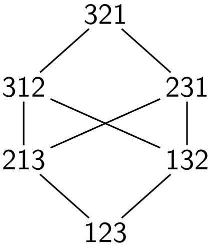

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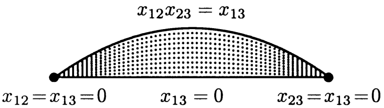

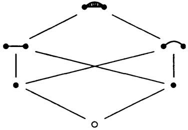

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Totally nonnegative Grassmannian is a regular CW complex

Theorem (Galashin–Karp–L. (2019))

The totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$ is a regular CW complex. In particular, every closed positroid cell $\Pi_{f,\geq 0}$ is homeomorphic to a closed ball.

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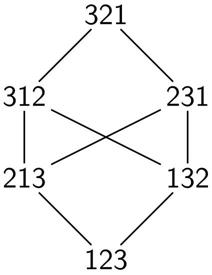
Conjecture: $\overline{\mathcal{E}}_n$ and $\overline{\mathcal{X}}_n$ are regular CW complexes.

Totally nonnegative flag variety



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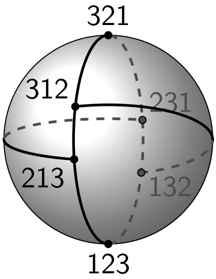
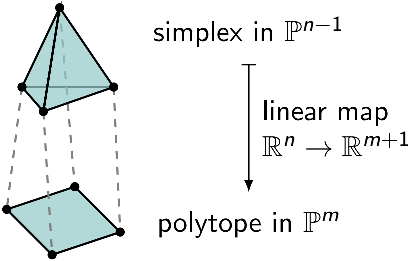


$\mathfrak{S}_3$ (Bruhat order)

The cells of the totally nonnegative flag variety for SL_3 are indexed by *intervals* in the Bruhat order of $\mathfrak{S}_3$.

Grassmann polytopes

- By definition, a polytope is the image of a simplex under an affine map:



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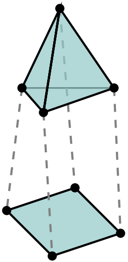

$$\begin{array}{ccc} \text{simplex in } \mathbb{P}^{n-1} & \dashrightarrow & \mathrm{Gr}_{k,n}^{\geq 0} \\ \Big\downarrow \text{linear map } \mathbb{R}^n \to \mathbb{R}^{m+1} & & \\ \text{polytope in } \mathbb{P}^m & & \end{array}$$

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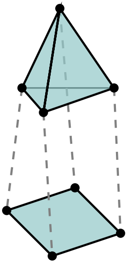

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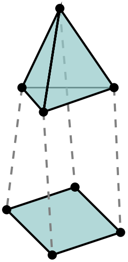

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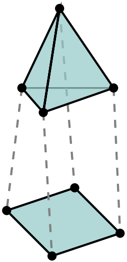

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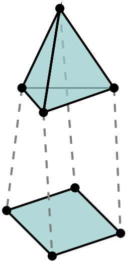

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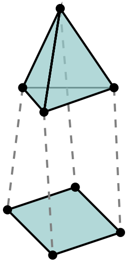
- Conjecture: Grassmann polytopes are regular CW complexes, and amplituhedra for m even are in addition homeomorphic to a ball. (Karp–Williams, Galashin–Karp–L., Blagojević–Galashin–Palić–Ziegler)

Triangulations of amplituhedra



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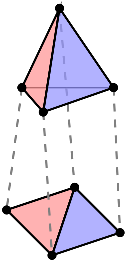

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cells of triangulation

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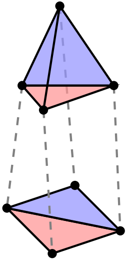

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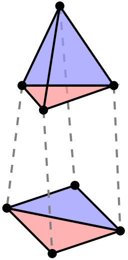

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Amplituhedra can be triangulated by (the images of) a collection of positroid cells $\Pi_{f,\geq 0}$.

- Karp–Williams: $m = 1$
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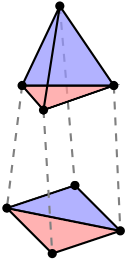
Triangulations of amplituhedra



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A triangulation of a Grassmann polytope gives a formula for its *canonical differential form*.

Positive geometries

Arkani-Hamed, Bai, L. (2017): a *positive geometry* is a space $X_{\geq 0}$ equipped with a meromorphic *canonical form* $\Omega(X_{\geq 0})$, with the property that

1. every boundary $C_{\geq 0}$ of $X_{\geq 0}$ is a positive geometry,
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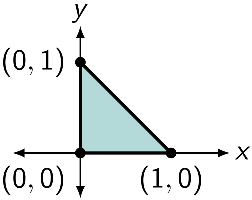

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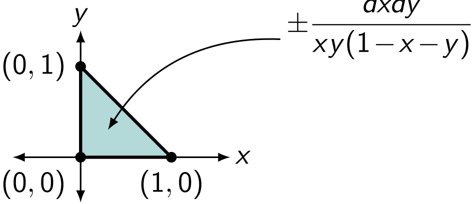

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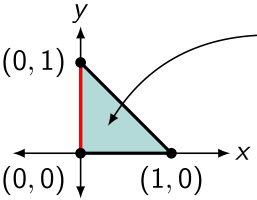

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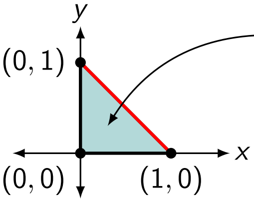

$$\pm \frac{dxdy}{xy(1-x-y)}$$

Arkani-Hamed, Bai, L. (2017): a *positive geometry* is a space $X_{\geq 0}$ equipped with a meromorphic *canonical form* $\Omega(X_{\geq 0})$, with the property that

1. every boundary $C_{\geq 0}$ of $X_{\geq 0}$ is a positive geometry,
2. $\Omega(X_{\geq 0})$ has simple poles along (the Zariski closure of) each $C_{\geq 0}$, and no other poles,
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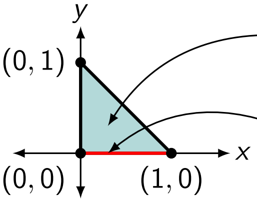

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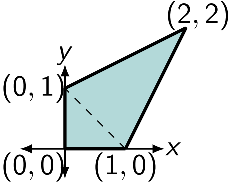

$$\Omega(P) = \frac{dxdy}{xy(1-x-y)} + \frac{-9dxdy}{(1-x-y)(2x-y-2)(2y-x-2)}$$
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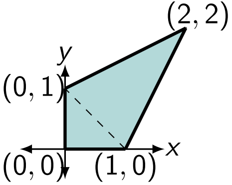

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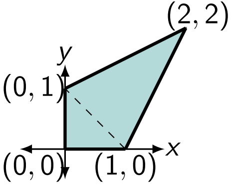

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- Other examples: $\mathrm{Gr}_{\geq 0}(k,n)$, $\Pi_{f,\geq 0}$, the nonnegative part of a toric variety, $M_{0,n}(\mathbb{R})_{\geq 0}$, $(G/B)_{\geq 0}$

Amplituhedron form and scattering amplitudes

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- The data of the momentum four-vectors of n particles is stored inside the $n \times (k + 4)$ Z-matrix defining the amplituhedron. The extra k-dimensions keep track of supersymmetry.

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Theorem (consequence of generalized Poincaré conjecture)

Suppose that X is a compact topological manifold with boundary, whose interior X° is contractible and whose boundary ∂X is homeomorphic to a sphere. Then X is homeomorphic to a closed ball.

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- Williams (2007): The face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is thin and shellable, so it is the face poset of a sphere. By Björner (1984), the homeomorphism type of a regular CW complex is determined by its face poset. Therefore by induction, ∂X is homeomorphic to a sphere.

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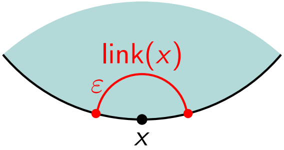
and furthermore that the link is itself a closed ball.



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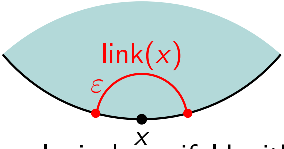

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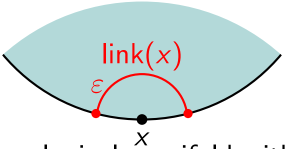

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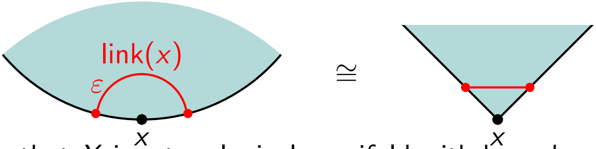

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Affine Bruhat atlas

- At the heart of Fomin and Shapiro's approach are factorization isomorphisms (first considered by Kazhdan and Lusztig)

$$C_u \longrightarrow X_u \times X^u,$$

one for each $u \in S_n$, where

1. C_u is a rotated open Schubert cell centred at u in the flag variety,
2. X_u and X^u are Schubert and opposite Schubert cells in the flag variety.

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There is no good choice for u (or I) for an arbitrary positroid variety Π_f.

- (Combinatorial Bruhat atlas) Knutson–L.–Speyer (also He and L.): the partial order on Bound(k,n) embeds as a lower order ideal in the Bruhat order of the affine symmetric group $\tilde{S}_n$.

Affine Bruhat atlas

- Snider (2011): rotated open Schubert cell $C_I := \{\Delta_I \neq 0\} \subset \mathrm{Gr}(k,n)$
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- e.g. Let $I = \{1,3\}$ with $k = 2, n = 4$. Then Snider's embedding is

$$\begin{bmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{bmatrix} \quad \mapsto \quad \begin{array}{cccccccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\ \cdots & 0 & a & 0 & b & 1 & 0 & \cdots & & & & \\ & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\ & & \cdots & 0 & d & 0 & c & 1 & 0 & \cdots & & \\ & & & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \\ & & & & \cdots & 0 & a & 0 & b & 1 & 0 & \cdots \\ & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} .$$

- We obtain the conic structure near x by translating x to a 'hidden' point in $\widetilde{\mathrm{Fl}}_n$ in the same cell as x, which is not in the image of the map φ_I.

Further directions

- We conjecture that the regularity theorem holds for totally nonnegative parts of Kac-Moody flag varieties.
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- Prove triangulations of amplituhedra and Grassmann polytopes exist.
- Give explicit (and preferably triangulation independent) formulae for the canonical form of a Grassmann polytope.

Some references

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