AFFINE STANLEY SYMMETRIC FUNCTIONS

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Abstract. We define a new family $\tilde{F}_w(X)$ of generating functions for $w \in \tilde{S}_n$ which are affine analogues of Stanley symmetric functions. We establish basic properties of these functions such as their symmetry and conjecture certain positivity properties. As an application, we relate these functions to the k-Schur functions of Lapointe, Lascoux and Morse as well as the cylindric Schur functions of Postnikov.

In [Sta84], Stanley introduced a family $\{F_w(X)\}$ of symmetric functions now known as *Stanley symmetric functions*. He used these functions to study the number of reduced decompositions of permutations $w \in S_n$. Later, the functions $F_w(X)$ were found to be stable limits of Schubert polynomials. Another fundamental property of Stanley symmetric functions is the fact that they are Schur-positive ([EG, LS]).

This extended abstract describes work in progress on an analogue of Stanley symmetric functions for the affine symmetric group $\tilde{S}_n$ which we call *affine Stanley symmetric functions*. Our first main theorem is that these functions $\tilde{F}_w(X)$ are indeed symmetric functions. Most of the other main properties of Stanley symmetric functions established in [Sta84] also have analogues in the affine setting.

Our definition of affine Stanley symmetric functions is motivated by relations with two other classes of symmetric functions which have received attention lately. Lapointe, Lascoux and Morse [LLM] initiated the study of *k-Schur functions*, denoted $s_\lambda^{(k)}(X)$, in their study of Macdonald polynomial positivity. Lapointe and Morse have more recently connected k-Schur functions with the Verlinde algebra of $SL(n)$. Separately, *cylindric Schur functions* were defined by Postnikov [Pos] in connection with the quantum cohomology of the Grassmannian (see also [GK]). We shall connect these two classes of symmetric functions via affine Stanley symmetric functions. More precisely, we show that when $w \in \tilde{S}_n$ is a "Grassmannian" affine permutation then $\tilde{F}_w(X)$ is "dual" to the k-Schur functions $s_\lambda^{(k)}(X)$. We call these functions $\tilde{F}_w(X)$ *affine Schur functions*. Affine Schur functions were earlier defined by Lapointe and Morse who called them *dual k-Schur functions*. In analogy with the usual Stanley symmetric function case, conjecture that all affine Stanley symmetric functions expand positively in terms of affine Schur functions. We then show that cylindric Schur functions are special cases of *skew affine Schur functions* and correspond to 321-avoiding affine permutations.

The non-affine case suggests that our work may be connected with the affine flag variety and objects that might be called "affine Schubert polynomials". Shimozono has conjectured a precise relationship between k-Schur functions and the homology of the affine Grassmannian. The dual conjecture ([MS]) is that affine Schur functions represent Schubert classes in the cohomology $H^*(\mathcal{G}/\mathcal{P})$ of the affine Grassmannian.

In section 1, we establish some notation for permutations and affine permutations, and for symmetric functions. In section 2 we recall the definition of Stanley symmetric functions, give their main properties and explain the relationship with Schubert polynomials. In section 3, we define affine Stanley symmetric functions and prove that they are symmetric. In section 4,

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we give basic properties of affine Stanley symmetric functions, imitating the results of [Sta84]. In section 5, we define affine Schur functions and relate them to k-Schur functions. In section 6, we connect skew affine Schur functions with cylindric Schur functions. In section 7, we make a number of positivity conjectures concerning the expansion of affine Stanley symmetric functions in terms of affine Schur functions. Finally, in section 8, we discuss relations with the affine flag variety and a generalisation to affine stable Grothendieck polynomials.

We should remark that Stanley symmetric functions for the hyperoctahedral group have also been defined; see [LTK].

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1. Preliminaries

1.1. **Symmetric group.** A positive integer $n \geq 2$ will be fixed throughout the paper. Let $\tilde{S}_n$ denote the affine symmetric group with simple generators $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\ s_i^2 &= 1 && \text{for all } i \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Here and elsewhere, the indices will be taken modulo n without further mention. There are many different explicit constructions of $\tilde{S}_n$, see for example [BB]. The symmetric group S_n embeds in $\tilde{S}_n$ as the subgroup generated by $s_1, s_2, \ldots, s_{n-1}$.

For an element $w \in \tilde{S}_n$ let $R(w)$ denote the set of *reduced words* for w. A word $\rho = (\rho_1 \rho_2 \cdots \rho_l) \in [0, n-1]^l$ is a reduced word for w if $w = s_{\rho_1} s_{\rho_2} \cdots s_{\rho_l}$ and l is the smallest possible integer for such a decomposition exists. The integer $l = l(w)$ is called the *length* of w. If $\rho, \pi \in R(w)$ for some w, then we write $\rho \sim \pi$. If ρ is an arbitrary word with letters from $[0, n-1]$ then we write $\rho \sim 0$ if it is not a reduced word of any affine permutation. If $w, u \in \tilde{S}_n$ then we say that w *covers* u and write $w \gtrdot u$ if $w = s_i \cdot u$ and $l(w) = l(u) + 1$. The transitive closure of $\gtrdot$ is called the *weak Bruhat order* and denoted $\geqslant$.

1.2. **Symmetric functions.** We will follow mostly [Mac, Sta99] for our symmetric function notation. Let Λ denote the ring of symmetric functions. Usually, our symmetric functions will have variables $x_1, x_2, \ldots$ and will be written as $f(x_1, x_2, \ldots)$ or $f(X)$. If we need to emphasize the variable used, we write Λ_X. We use λ, μ and ν to denote partitions. We will use m_λ, p_λ, e_λ, h_λ and s_λ to denote the *monomial*, *power sum*, *elementary*, *homogeneous* and *Schur* bases of Λ. Let $\langle ., . \rangle$ denote the *Hall inner product* of Λ satisfying $\langle h_\lambda, m_\mu \rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. For $f \in \Lambda$, write $f^\perp : \Lambda \to \Lambda$ for the linear operator adjoint to multiplication by f with respect to $\langle ., . \rangle$. We let $\omega : \Lambda \to \Lambda$ denote the algebra involution of Λ sending h_n to e_n.

If $f(X) \in \Lambda$ then $f(X, Y) = \sum_i f_i(X) \otimes g_i(Y) \in \Lambda_X \otimes \Lambda_Y$ for some f_i and g_i. This is the *coproduct* of f, written $\Delta f = \sum_i f_i \otimes g_i \in \Lambda \otimes \Lambda$. We have the following formula for the coproduct ([Mac]):

$$\Delta f = \sum_\lambda s_\lambda^\perp f \otimes s_\lambda. \tag{1}$$

Let $\mathcal{P}ar^n$ denote the set $\{\lambda \mid \lambda_1 \leq n-1\}$ of partitions with no row longer than $n-1$. The following two subspaces of Λ will be important to us:

$$\Lambda^{(n)} = \mathbb{C}\left\langle m_\lambda \mid \lambda \in \mathcal{P}ar^n \right\rangle$$
$$\Lambda_{(n)} = \mathbb{C}\left\langle h_\lambda \mid \lambda \in \mathcal{P}ar^n \right\rangle = \mathbb{C}\left\langle e_\lambda \mid \lambda \in \mathcal{P}ar^n \right\rangle = \mathbb{C}\left\langle p_\lambda \mid \lambda \in \mathcal{P}ar^n \right\rangle.$$

If $f \in \Lambda_{(n)}$ and $g \in \Lambda^{(n)}$ then define $\langle f, g\rangle$ to be their usual Hall inner product within Λ. Thus $\{h_\lambda\}$ and $\{m_\lambda\}$ with $\lambda \in \mathcal{P}ar^n$ form dual bases of $\Lambda_{(n)}$ and $\Lambda^{(n)}$. Note that $\Lambda_{(n)}$ is a subalgebra of Λ but $\Lambda^{(n)}$ is not closed under multiplication. Instead, $\Lambda^{(n)}$ is a coalgebra; it is closed under comultiplication.

2. Stanley symmetric functions

Let $w \in S_n$ with length $l = l(w)$. Define the generating function $F_{w^{-1}}(X)$ by

$$F_{w^{-1}}(x_1, x_2, \ldots) = \sum_{a_1 a_2 \cdots a_l \in R(w)} \sum_{\substack{1 \leq b_1 \leq b_2 \leq \cdots \leq b_l \\ a_i > a_{i+1} \Rightarrow b_{i+1} > b_i}} x_{b_1} x_{b_2} \cdots x_{b_l}.$$

We have indexed the $F_{w^{-1}}(X)$ by the inverse permutation to agree with the definition we shall give later. These generating functions, known as *Stanley symmetric functions*, were shown in [Sta84] to be symmetric. Stanley also studied these functions under the action of ω, the action of $s_1^\perp$ and also proved that the Schur expansions of $F_w(X)$ possess *dominant* terms. Edelman and Greene [EG] and Lascoux and Schützenberger [LS] showed that Stanley symmetric functions are Schur positive so that if

$$F_w(X) = \sum_\lambda a_{w\lambda} s_\lambda(X)$$

then $a_{w\lambda} \geq 0$. Note that the length $l(w)$ is equal to the degree of F_w and the number $|R(w)|$ of reduced decompositions of w is given by the coefficient of $x_1 x_2 \cdots x_l$ in F_w. We now give a different formulation of the definition in a manner similar to [FS].

Let $\mathbb{C}[S_n]$ denote the group algebra of the symmetric group equipped with a inner product $\langle w, v\rangle = \delta_{wv}$. Define linear operators $u_i : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ for $i \in [1, n-1]$ by

$$u_i.w = \begin{cases} s_i.w & \text{if } l(s_i.w) > l(w), \\ 0 & \text{otherwise.} \end{cases}$$

The operators satisfy the braid relations $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ together with $u_i^2 = 0$ and $u_i u_j = u_j u_i$ for $|i-j| \geq 2$. They generate an algebra known as the *nilCoxeter algebra*. Note that the action on $\mathbb{C}[S_n]$ is a faithful representation of these relations.

Let $A_k(u) = \sum_{b_1 > b_2 > \cdots > b_k} u_{b_1} u_{b_2} \cdots u_{b_k}$. Then the Stanley symmetric functions can be written as

$$F_w(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle A_{a_t}(u) A_{a_{t-1}}(u) \cdots A_{a_1}(u) \cdot 1, w\right\rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t} \tag{2}$$

where the sum is over all compositions a.

For completeness, we explain briefly the relationship between $F_w(X)$ and the Schubert polynomials of Lascoux and Schützenberger. For $w \in S_n$, we have a Schubert polynomial $\mathfrak{S}_w \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. If $w \in S_n$, then $w \times 1^s \in S_{n+s}$ denotes the corresponding permutation of S_{n+s} acting on the elements $[1, n]$ of $[1, n+s]$. Similarly, $1^s \times w \in S_{n+s}$ denotes the corresponding permutation acting on the elements $[s+1, n+s]$ of $[1, n+s]$. Schubert polynomials have the important stability property $\mathfrak{S}_w = \mathfrak{S}_{w \times 1^s}$. Stanley symmetric functions

$F_w(X)$ are obtained by taking the other limit: $F_w = \lim_{s\to\infty} \mathfrak{S}_{1^s\times w}$. The limit is taken by treating both sides as formal power series and taking the limit of each coefficient.

3. Affine Stanley symmetric functions

Our first definition of affine Stanley symmetric functions will imitate the definition (2) above. Let $\mathcal{U}_n$ be the *affine nilCoxeter algebra* generated over $\mathbb{C}$ by generators $u_0, u_1, \ldots, u_{n-1}$ satisfying

$$\begin{aligned} u_i^2 &= 0 && \text{for all } i \in [0,n], \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} && \text{for all } i \in [0,n], \\ u_i u_j &= u_j u_i && \text{for all } i,j \in [0,n] \text{ satisfying } |i-j| \geq 2. \end{aligned}$$

Here and henceforth the indices are to be taken modulo n. A basis of $\mathcal{U}_n$ is given by the elements $u_w = u_{\rho_1} u_{\rho_2} \cdots u_{\rho_l}$ where $\rho = (\rho_1 \rho_2 \cdots \rho_l)$ is some reduced word for w.

Define $h_k(\mathbf{u}) \in \mathcal{U}_n$ for $k \in [0, n-1]$ by

$$h_k(\mathbf{u}) = \sum_{A \in \binom{[0,n-1]}{k}} u_A$$

where for a k-subset $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$ the element $u_A \in \mathcal{U}_n$ is defined as any expression $u_{a_1} u_{a_2} \cdots u_{a_k}$ where if i and $i+1$ (modulo n) are both in A then u_{i+1} must precede u_i. All such expressions are equal within $\mathcal{U}_n$. For example if $n = 9$ and $A = \{0, 2, 4, 5, 6, 8\}$ then $u_A = u_0 u_8 u_2 u_6 u_5 u_4 = u_2 u_6 u_5 u_4 u_0 u_8 = \cdots$. A similar definition of $h_k(\mathbf{u})$ was given by Postnikov [Pos], in the context of the *affine nil-Temperley-Lieb algebra*.

Define a representation of $\mathcal{U}_n$ on $\mathbb{C}[\tilde{S}_n]$ by

$$u_i.w = \begin{cases} s_i.w & \text{if } l(s_i.w) > l(w), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this is indeed a representation of $\mathcal{U}_n$. Equip $\mathbb{C}[\tilde{S}_n]$ with the inner product $\langle w, v\rangle = \delta_{wv}$. The following definition was heavily influenced by [FG].

Definition 1. Let $w \in \tilde{S}_n$. Define the *affine Stanley symmetric functions* $\tilde{F}_w(X)$ by

$$\tilde{F}_w(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle h_{a_t}(u) h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot 1, w \right\rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t},$$

where the sum is over compositions of $l(w)$ satisfying $a_i \in [0, n-1]$.

The seemingly more general "skew" affine Stanley symmetric functions

$$\tilde{F}_{w/v}(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle h_{a_t}(u) h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot v, w \right\rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$$

are actually equal to the usual affine Stanley symmetric functions $\tilde{F}_{wv^{-1}}(X)$.

Our first proposition follows from the definition.

Proposition 2. *Suppose $w \in S_n \subset \tilde{S}_n$. Then $\tilde{F}_w(X) = F_w(X)$.*

The main theorem of this section is the following.

Theorem 3. *The generating functions $\tilde{F}_w(X)$ are symmetric.*

Theorem 3 follows immediately from Proposition 5. In the following, intervals $[a, b]$ are to be taken in the cyclic fashion within $[0, n-1]$. Also, max and min of a cyclic interval is meant to be taken modulo n in the obvious manner. So if $n = 6$ then $[4, 1] = \{4, 5, 0, 1\}$ and $\max([4, 1]) = 1$ and $\min([4, 1]) = 4$. We will need a technical lemma first.

Lemma 4. *We have the following identities for reduced words.*

(1) *Let $a, b \in [0, n-1]$ with $a \neq b-1$. Then $a(a-1)(a-2)\cdots ba(a-1)(a-2)\cdots b \sim 0$.*
(2) *Let $a, b, c \in [0, n-1]$ satisfying $a \neq b-1$; $c \neq b$ and $c \in [b, a]$. Then $a(a-1)(a-2)\cdots bc \sim (c-1)a(a-1)(a-2)\cdots b$.*

Proof. Both results can be calculated by induction. □

Proposition 5. *The elements $h_k(\mathbf{u})$ for $k \in [0, n-1]$ commute.*

Proof. For each $w \in \tilde{S}_n$, we calculate the coefficient of u_w in $h_x(\mathbf{u})h_y(\mathbf{u})$ and $h_y(\mathbf{u})h_x(\mathbf{u})$. We assume that x and y are both not equal to 0 for otherwise the result is obvious. Let $u_w = u_A u_B$ where $|A| = x$ and $|B| = y$. We need to exhibit a bijection between reduced decompositions of this form and those of the form $u_w = u_C u_D$ with $|C| = y$ and $|D| = x$. We assume for simplicity (though it is not crucial to the proof) that $A \cup B = [0, n-1]$ for otherwise we are in the non-affine case and the proposition follows from results of Stanley [Sta84] or Fomin-Greene [FG]. Let $A = \bigcup_i A_i$ and $B = \bigcup_i B_i$ be minimal decompositions of A and B into cyclic intervals. If $A_i \subset B_j$ for some pair (i, j) then we call A_i an *inner* interval and similarly for $B_k \subset A_l$. Otherwise the interval is called *outer*.

Using Lemma 4 and our assumption that $A \cup B = [0, n-1]$ we can describe the outer intervals in an explicit manner. Each outer interval A_i *touches* an outer interval $\mathrm{rn}(A_i) = B_k$ called the *right neighbour* of A_i, for a unique k, so that $\min(A_i) = \max(B_k) + 1$. Also A_i overlaps with an outer interval $\mathrm{ln}(A_i) = B_l$ for a unique l, so that $\max(A_i) \geq \min(B_l) - 1$ called the *left neighbour*. If $\mathrm{rn}(A_i) = B_k$ then we also write $A_i = \mathrm{ln}(B_k)$ and similarly for $\mathrm{rn}(B_k)$. Note that it is possible that $\mathrm{rn}(A_i) = \mathrm{ln}(A_i)$ since we are working cyclically.

Our bijection will depend only locally on each pair of an outer interval A^* and its right neighbour $B^* = \mathrm{rn}(A')$. We call the interval $I = [\min(B^*), \min(\mathrm{ln}(A')) - 1]$ a *critical interval*. Critical intervals cover $[0, n-1]$ in a disjoint manner. For example, suppose $n = 10$ and $A^* = \{1, 2, 3, 6, 7, 8, 9\}$ and $B^* = \{0, 2, 4, 5, 7, 9\}$ (Figure 1), so that $u_A u_B = u_9u_8u_7u_6u_3u_2u_1u_0u_9u_7u_5u_4u_2u_0$. Then $A_1 = [1, 3]$ and $A_2 = [6, 9]$ are both outer intervals. Also $B_1 = [2, 5]$, $B_2 = \{7\}$ and $B_3 = [9, 0]$. Only B_2 is an inner interval. The left neighbour of A_1 is $\mathrm{ln}(A_1) = B_1$ and the right neighbour is $\mathrm{rn}(A_1) = B_3$. The critical intervals are $[1, 9]$ and $[2, 8]$.

Let $a = \min(\mathrm{ln}(A')) - 1$ and $b = \min(B^*) + 1$. Let $c = |[b, a]|$, $d = |A \cap [b, a]|$ and $e = |B \cap [b, a]|$. Renaming for convenience, we let $S_1, S_2, \ldots, S_s$ be the inner intervals (of B) contained in A^* and $T_1, \ldots, T_t$ be those contained in B^*, arranged so that $S_k > S_{k+1}$ for all k within $[b, a]$ and similarly $T_k > T_{k+1}$. We now define a subset $U \subset [b, a]$ satisfying $|U| = d$. The algorithm begins with $U = [b, a]$ and a changing index i set to $i := a$ to begin with. The index i decreases from a to b and at each step the element i may be removed from U according to the rule:

(1) If $i \in A^*$ then we remove it from U unless $i \in S_k$ for some $k \in [1, s]$.
(2) If $i \in B^*$ then we remove it from U unless $i \in T_k + 1$ for some $k \in [1, s]$.
(3) Otherwise we do not remove i from U and set $i := i - 1$. Repeat.

When $|U| = d$ we stop the algorithm. The algorithm always terminates with $|U| = d$ since there are at least $c - d = |[b, a]| - (A \cap [b, a])$ elements to remove. In fact the algorithm terminates before we reach $\max(\mathrm{rn}(B^*))$. We will denote the result of the algorithm by $\phi(A^* \cup_i T_i, B^* \cup_i S_i) = U$.

The bijection $u_Au_B \mapsto u_Cu_D$ is obtained by letting $D \subset [0, n-1]$ be the subset obtained from B by changing $B \cap I$ in each critical interval I to U. By the definition of U we see that $|D| = |A|$. We claim that $u_Au_B(u_D)^{-1} = u_C$ for some C satisfying $|C| = |B|$. We can calculate this locally on each critical interval since the $u_{D\cap I}$ commute as I varies over critical intervals. Note that U always has the form $S_1 \cup S_2 \cup \cdots \cup S_{s'} \cup [b, a']$ for some $s' \leq s$ where $a' > \max(B^*)$ or the form $S_1 \cup \cdots \cup S_s \cup \{T_1 + 1\} \cup \{T_2 + 1\} \cup \cdots \cup \{T_{t'} + 1\} \cup [b, a']$ where $a' \leq \max(B^*)$. In the following, let $u' = u^{-1}$.

Let us assume that U has the first form. Focusing on $I = [b, a] = [\min(B^*), \min(\ln(A^*)) - 1]$ we are interested in

$$\underline{u} = u_{A^*\cap I}u_{T_1} \cdots u_{T_t}u_{S_1} \cdots u_{S_s}u_{B^*}u'_{[b,a']}u'_{S_{s'}} \cdots u'_{S_1}.$$

Then we get

$$\begin{aligned}\underline{u} &= u_{A^*\cap I}u_{T_1} \cdots u_{T_t}u_{S_{s'+1}} \cdots u_{S_s}u'_{[\max(B^*)+1,a']}\\ &= u_{S_{s'+1}-1} \cdots u_{S_s-1}u_{T_1} \cdots u_{T_t}u_{A^*}u'_{[\max(B^*)+1,a']}\\ &= u_{S_{s'+1}-1} \cdots u_{S_s-1}u_{T_1} \cdots u_{T_t}u_{[a'+1,a]} \qquad \text{using } \max(B^*) + 1 = \min(A^*).\end{aligned}$$

We used Lemma 4 repeatedly and also the fact that the certain intervals do not "touch" and so commute. Set $U' = [a' + 1, a] \cup \{S_{s'+1} + 1\} \cup \cdots \cup \{S_s + 1\} \cup T_1 \cup \cdots T_t$. The other form of U has a similar calculation. One checks that we can combine this argument for each critical interval showing that $u_Au_B(u_D)^{-1}$ is indeed equal to u_C for some C.

Finally, we need to show that this map is a bijection. Again we work locally on a critical interval and assume that U has the first form. If we replace A^* by U' and B^* by U, then our internal intervals are $S'_1 = S_1, \ldots S'_{s'} = S_{s'}$ and $T'_1 = S_{s'+1} + 1, \ldots, T'_{s-s'} = S_{s'} + 1, T'_{s-s'+1} = T_1, \ldots T'_{s-s'+t} = T_t$. We now show that $B^* \cup_i S_i = \phi(U', U)$ from which the bijectivity will follow. By definition $\phi(U', U)$ keeps $S'_1, S'_2, \ldots$ and keeps $T'_1 - 1, T'_2 - 1, \ldots, T'_{s-s'} - 1$, removing all other values up to this point. We see that we obtain $B^* \cup_i S_i$ back in this way. A similar argument works for the second form. $\square$

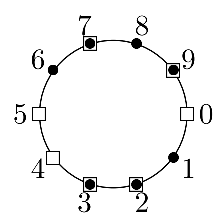

FIGURE 1. Dots represent elements of A^* squares represent elements of B^*.

Example 1. We illustrate the map $U = \phi(A^* \cup_i T_i, B^* \cup_i S_i)$ of the proof. Suppose $[b, a] = [2, 20]$ and $A^* = [14, 20]$, $B^* = [2, 13]$. Let $S_1 = [16, 18]$ and $T_1 = [8, 11]$ and $T_2 = \{5\}$ be the inner intervals. Then $d = 12$ and $U = \{2, 3, 4, 5, 6, 9, 10, 11, 12, 16, 17, 18\}$. We can compute that

$$u_{A^*}u_{11}u_{10}u_9u_8u_5u_{B^*}u_{18}u_{17}u_{16}u_2u_3u_4u_5u_6u_9u_{10}u_{11}u_{12}u_{16}u_{17}u_{18} = u_{A^*}u_{[7,13]}u_5$$

so that $U' = [20, 7] \cup \{5\}$. Finally one checks that $B^* \cup_i S_i = \phi(U', U)$.

We end this section by giving an alternative description of the affine Stanley symmetric functions. Let $w \in \tilde{S}_n$. Let $a = (a_1, \ldots, a_l) \in R(w)$ be a reduced word and $b = (b_1 \geq b_2 \cdots \geq b_l)$ be an positive integer sequence. Then (a, b) is called a *compatible pair for w* if whenever $b_i = b_{i+1} = \cdots = b_j$ and $\{k, k+1\} \subset \{a_i, a_{i+1}, \ldots, a_j\}$ then we have that $k + 1$ precedes k

(for any i, j, k). Two compatible pairs (a, b) and (a', b') are equivalent if $b = b'$ and for any maximal interval $[i, j] \subset [1, l]$ satisfying $b_i = b_{i+1} = \cdots = b_j$ we have that $a_i a_{i+1} \cdots a_j$ and $a'_i a'_{i+1} \cdots a'_j$ are reduced words for the same affine permutation. Then

$$F_w(X) = \sum_{\overline{(a,b)}} x_{b_1} x_{b_2} \cdots x_{b_l}$$

where the sum is over equivalence classes $\overline{(a, b)}$ of compatible pairs for w.

4. Basic properties

We give a number of basic properties for the functions $\tilde{F}_w$. The first main property follows immediately from the definition.

Proposition 6. *We have $\tilde{F}_w \in \Lambda^{(n)}$ for each $w \in \tilde{S}_n$.*

In fact we shall see later that they span the subspace $\Lambda^{(n)}$.

Theorem 7 (Coproduct formula)**.** *The following coproduct expansion holds:*

$$\tilde{F}_w(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{uv=w} \tilde{F}_v(x_1, x_2, \ldots)\tilde{F}_u(y_1, y_2, \ldots).$$

In particular we have

$$s_1^{\perp} \tilde{F}_w = \sum_{w \gtrdot v} \tilde{F}_v.$$

Proof. The first formula follows immediately from the definition and the fact that $\tilde{F}_{w/v}(Y) = \tilde{F}_{wv^{-1}}(Y)$. To obtain the second formula, we first write, using the first formula and (1),

$$\sum_{uv=w} \tilde{F}_v(X) \otimes \tilde{F}_u(Y) = \sum_{\lambda} s_\lambda^{\perp} \tilde{F}_w(X) \otimes s_\lambda(Y).$$

The terms of the formula are to be interpreted within Λ, even though the sum is an element of $\Lambda^{(n)}$. Now take the inner product of both sides with $s_1(Y)$ to get

$$s_1^{\perp} \tilde{F}_w(X) = \sum_{uv=w} \tilde{F}_v(X) \left\langle \tilde{F}_u(Y), s_1(Y) \right\rangle.$$

Now $\left\langle \tilde{F}_u(Y), s_1(Y) \right\rangle = 0$ unless $u = s_i$ is a simple reflection for some i, in which case $\tilde{F}_{s_i}(Y) = s_1(Y)$. This gives the second formula. □

Define $\omega : \Lambda_{(n)} \to \Lambda_{(n)}$ as usual by $\omega : h_i \mapsto e_i$. Define $\omega^+ : \Lambda^{(n)} \to \Lambda^{(n)}$ by requiring that $\langle \omega(f), \omega^+(g) \rangle = \langle f, g \rangle$. Alternatively, we require that $\{e_\lambda \mid \lambda \in \mathcal{P}ar^n\}$ and $\{\omega^+(m_\lambda) \mid \lambda \in \mathcal{P}ar^n\}$ form dual bases. The map ω^+ is clearly an involution but it does not agree with ω (see for example [Sta99, Chapter 7, Ex. 9]).

Denote by $w \mapsto w^*$ the involution of $\tilde{S}_n$ given by $s_i \mapsto s_{n-1-i}$.

Theorem 8 (Conjugacy formula)**.** *Let $w \in \tilde{S}_n$. Then $\omega^+(\tilde{F}_w) = \tilde{F}_{w^*}$.*

We shall prove Theorem 8 by calculating within the subalgebra $\Lambda_{(n)}(\mathbf{u})$ of $\mathcal{U}_n$ generated by $\{h_k(\mathbf{u})\}$. By Proposition 5, this subalgebra is naturally the homomorphic image to $\Lambda_{(n)}$. In fact $\Lambda_{(n)}(\mathbf{u})$ is abstractly isomorphic to $\Lambda_{(n)}$. For an element $f \in \Lambda_{(n)}$, we let $f(\mathbf{u})$ denote the corresponding image in $\Lambda_{(n)}(\mathbf{u})$.

Theorem 9. *The elements $e_k(\mathbf{u}) \in \mathcal{U}_n$ are given by*

$$e_k(\mathbf{u}) = \sum_{A \in \binom{[0,n-1]}{k}} \tilde{u}_A,$$

where for a k-subset $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$ the element $\tilde{u}_A \in \mathcal{U}_n$ is defined as any expression $u_{a_1} u_{a_2} \cdots u_{a_k}$ where if i and $i+1$ (modulo n) are both in A then u_i must precede u_{i+1} within $\tilde{u}_A$.

Sketch of proof. We verify this using the relation $e_k = h_k - h_{k-1}e_1 + \cdots \pm h_1 e_{k-1}$. Since $k \leq n$ we can restrict our attention to proper subsets of $[0, n-1]$, one at a time. After this, the theorem follows by induction. $\square$

Thus $u_i \mapsto u_{n-1-i}$ is an involution of $\mathcal{U}_n$ sending $h_k(\mathbf{u})$ to $e_k(\mathbf{u})$. More generally, when λ is a hook so that $s_\lambda \in \Lambda_{(n)}$ then $s_\lambda(\mathbf{u})$ can be written as a sum over the reading words of certain tableaux (see [Lam]). We shall not need this generality, however see Conjecture 15.

Theorem 8 follows by writing

$$\Omega^{(n)} = \sum_{\lambda \in \mathcal{P}ar^n} h_\lambda(\mathbf{u}) m_\lambda = \sum_{\lambda \in \mathcal{P}ar^n} e_\lambda(\mathbf{u}) w^+(m_\lambda)$$

and using the fact that $\tilde{F}_w(X) = \langle \Omega^{(n)} \cdot 1, w \rangle$.

5. Affine Schur functions

We now describe another representation of $\tilde{S}_n$ and $\mathcal{U}_n$. Let $\mathcal{P}$ denote the set of doubly infinite $(0,1)$-sequences $p = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$ and let $\mathbb{C}[\mathcal{P}]$ denote the space of formal $\mathbb{C}$- linear combinations of such sequences. Let $\tilde{S}_n$ act on $\mathcal{P}$ by letting s_i act on $p = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$ by swapping p_{kn+i} and p_{kn+i+1} for each $k \in \mathbb{Z}$. One can check directly that this defines a representation of $\tilde{S}_n$.

A subrepresentation $\mathbb{C}[\mathcal{P}^*]$ of $\mathbb{C}[\mathcal{P}]$ is given by taking only those bit sequences p satisfying $p_N = 1$ for sufficiently small $N << 0$ and $p_N = 0$ for $N >> 0$. These sequences correspond to the *edge sequences* of a partition (see [Sta99, vL]). The edge sequence is obtained by drawing the partition in the English notation and reading the "edge" of the partition from bottom left to top right – writing a 1 if you go up and writing a 0 if you go to the right. Let $p(\lambda)$ denote the edge sequence associated to λ normalised so that $p(\emptyset)_i = 1$ for $i \leq 0$ and $p(\emptyset)_i = 0$ for $i \geq 1$. For example, $p(32) = (\ldots, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, \ldots))$. It is easy to see that $\mathbb{C}[\mathcal{P}^*]$ is indeed a subrepresentation, but it is by no means irreducible. Let $\mathcal{P}^n$ denote the set $\left\{\tilde{S}_n \cdot (\emptyset)\right\}$ of the orbit of the edge sequence of the empty partition. This orbit can be described very naturally when thought of as partitions: it is the set of n-cores [Las]. From now on we will identify $\mathcal{P}^n$ with the set of n-cores. Since the stabiliser of the empty partition is $S_n \subset \tilde{S}_n$, the set $\mathcal{P}^n$ of n-cores is in fact isomorphic to $\tilde{S}_n / S_n$ where here S_n is generated by $s_1, s_2, \ldots, s_{n-1}$.

Now let $\mathcal{U}_n$ act on $\mathbb{C}[\mathcal{P}^n]$ by

$$u_i \cdot \nu = \begin{cases} s_i \cdot \nu & \text{if } s_i \cdot \nu \text{ is obtained from } \nu \text{ by adding boxes.} \\ 0 & \text{otherwise.} \end{cases}$$

One checks ([Las, LM]) that $s_i \cdot \nu$ is always obtained from ν by either adding boxes or removing boxes (never both) when $\nu \in \mathcal{P}^n$. The fact that this defines an action of $\mathcal{U}_n$ is easy to verify. Equip $\mathcal{P}^n$ with the inner product $\langle \nu, \mu \rangle = \delta_{\nu\mu}$.

Definition 10. The *skew affine Schur functions* $\tilde{F}_{\nu/\mu}(X)$ are given by

$$\tilde{F}_{\nu/\mu}(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle h_{a_t}(u)h_{a_{t-1}}(u)\cdots h_{a_1}(u)\cdot\mu,\nu\right\rangle x_1^{a_1}x_2^{a_2}\cdots x_t^{a_t}.$$

The *affine Schur functions* are given by $\tilde{F}_\nu(X) = \tilde{F}_{\nu/\emptyset}(X)$.

By Proposition 5, these functions are actually symmetric. One can view affine Schur functions as the generating functions for certain semistandard tableaux built on n-cores. These tableaux are called "k-tableaux" by Lapointe and Morse [LM]. Affine Schur functions had earlier been defined by Lapointe and Morse, and were called *dual k-Schur functions*.

The following proposition is immediate.

Proposition 11. *If $w\cdot\emptyset = \nu \in \mathcal{P}^n$ and w is a minimum length representative in its coset of $\tilde{S}_n/S_n$ then*

$$\tilde{F}_\nu(X) = \tilde{F}_w(X)$$

so that affine Schur functions are special cases of affine Stanley symmetric functions. We write $w = w(\nu)$.

We will call affine permutations of the proposition *Grassmannian*. Note that all the weak Bruhat orders corresponding to $\tilde{S}_n$ modulo a maximal parabolic subgroup are isomorphic so that we lose no generality considering only this maximal parabolic subgroup. This is unlike the non-affine case, where Grassmannian permutations for different maximal parabolics are significantly different.

Theorem 12. *The affine Schur functions form a basis for $\Lambda^{(n)}$.*

We sketch a proof of this theorem. In fact we show that the transition matrix between affine Schur functions and monomial symmetric functions is unitriangular. A more general statement is true for affine Stanley symmetric functions: the monomial (and also affine Schur) expansion contains a unique *dominant* term (see [Sta84] for the non-affine version of this result).

Affine Grassmannian permutations are also naturally indexed by partitions $\lambda \in \mathcal{P}ar^n$. An easy bijection is given by the *code* or *affine inversion table* [BB, Las]. This is a vector $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{N}^n - \mathbb{P}^n$ of non-negative entries with at least one 0. It is shown in [BB] that there is a bijection between codes and affine permutations. The action of the simple generator s_i on the code $c = (c_0, c_1, \ldots, c_{n-1})$ can be described by

$$s_i\cdot(c_0,\ldots,c_{i-1},c_i,\ldots,c_{n-1}) = (c_0,\ldots,c_i+1,c_{i-1},\ldots,c_{n-1})$$

whenever $c_i > c_{i-1}$. To each affine permutation w we let $\mu(w)$ denote the partition conjugate to the decreasing permutation of its code $c(w)$. Grassmannian permutations correspond to weakly increasing codes, so that $w \mapsto \mu(w)$ is a bijection for Grassmannian permutations. From hereon, we will label an affine Schur function with partitions $\lambda \in \mathcal{P}ar^n$, so that $\tilde{F}_\nu(X) = \tilde{F}_\lambda(X)$ if $\nu \in \mathcal{P}^n$ and $\lambda = \mu(w(\nu))$.

We observe that applying a term of $h_k(\mathbf{u})$ to w will increase k different entries of $c(w)$ by 1 (assuming the result is non-zero). For $w \in \tilde{S}_n$, let $\alpha_{w\lambda}$ be given by $\tilde{F}_w(X) = \sum_{\lambda\in\mathcal{P}ar^n}\alpha_{w\lambda}m_\lambda$.

We obtain the following theorem, which implies Theorem 12.

Theorem 13. *Let $w \in \tilde{S}_n$. Then*

- *If $\alpha_{w\lambda} \neq 0$ then $\lambda \preceq \mu(w)$.*
- *We have $\alpha_{w\mu(w)} = 1$.*

We now describe the relationship between affine Schur functions and k-Schur functions (with $k = n-1$). k-Schur functions were originally used to investigate Macdonald polynomial positivity and were defined as symmetric functions with coefficients in $\mathbb{C}(t)$. There are a number of different definitions of k-Schur functions [LLM, LM] which conjecturally agree. The form of the k-Schur functions that we will use are (conjecturally) the $t = 1$ specialisations of the original definition. Suppose $\tilde{F}_\lambda(X) = \sum_\mu K^{(n)}_{\lambda\mu} m_\mu$ where $\lambda \in \mathcal{P}ar^n$ and the sum is over μ satisfying $\mu \in \mathcal{P}ar^n$. Then the k-Schur functions $s^{(k)}_\lambda(X) \in \Lambda_{(n)}$ are given by requiring that

$$h_\mu = \sum_\lambda K^{(n)}_{\lambda\mu} s^{(k)}_\lambda(X).$$

A form of this definition is called the "k-Pieri" rule in [LM]. Affine Schur functions and k-Schur functions are dual in the sense that $\left\langle s^{(k)}_\mu, \tilde{F}_\nu \right\rangle = \delta_{\mu\nu}$. This can be seen by writing the *affine Cauchy kernel*

$$\begin{aligned}\Omega^{(n)} &= \sum_{\mu:\ \mu\in\mathcal{P}ar^n} h_\mu(X) m_\mu(Y) = \sum_{\mu:\ \mu\in\mathcal{P}ar^n} \left(\sum_{\lambda:\ \lambda\in\mathcal{P}ar^n} K^{(n)}_{\lambda\mu} s^{(k)}_\lambda(X) \right) m_\mu(Y) \\ &= \sum_{\lambda:\ \lambda\in\mathcal{P}ar^n} s^{(k)}_\lambda(X) \left(\sum_{\mu:\ \mu\in\mathcal{P}ar^n} K^{(n)}_{\lambda\mu} m_\mu(Y) \right) = \sum_{\lambda:\ \lambda\in\mathcal{P}ar^n} s^{(k)}_\lambda(X) \tilde{F}_\lambda(Y).\end{aligned}$$

6. Relation with cylindric Schur functions

We have seen that affine Schur functions correspond to affine Stanley symmetric functions for Grassmannian permutations.

Cylindric Schur functions [GK] are special cases of skew affine Schur functions. One can define them in the same way as skew affine Schur functions by letting $\mathcal{U}_n$ act on periodic bit sequences $p = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$ satisfying $p_i = p_{i+n}$. It is clear that periodic bit sequences are closed under the action of $\tilde{S}_n$ and in fact form $n+1$ finite orbits depending on the value of $p_1 + p_2 + \cdots + p_n \in [0, n]$. They can be thought of as edge sequences of partitions lying on a cylinder.

It is more convenient to work with cylindric partitions instead of the periodic edge sequences, and let the cylindric partitions be drawn on the plane (satisfying certain invariance conditions under translation) so that many cylindric partitions may have the same edge sequence but be considered distinct – we may translate a cylindric partition within the plane to obtain a different one. Formally, a cylindric partition λ is an infinite lattice path in $\mathbb{Z}^2$, consisting only of moves upwards and to the right, invariant under the translation by a vector $(k, n-k)$ for some $k \in [0, n]$. We denote the set of cylindric partitions by $\mathcal{P}^c$.

If λ is a cylindric partition then $s_i \cdot \lambda$ is the cylindric partition obtained from λ by either adding boxes at all corners along diagonals congruent to i mod n, or removing boxes, or doing nothing. Define $u_i : \mathbb{C}[\mathcal{P}^c] \to \mathbb{C}[\mathcal{P}^c]$ by

$$u_i \cdot \lambda = \begin{cases} s_i \cdot \lambda & \text{if } s_i \cdot \lambda \text{ is obtained from } \lambda \text{ by adding boxes.} \\ 0 & \text{otherwise.} \end{cases}$$

This defines a representation of $\mathcal{U}_n$ on $\mathbb{C}[\mathcal{P}^c]$, and equipping $\mathbb{C}[\mathcal{P}^c]$ with the natural inner product one can check that for $\lambda, \mu \in \mathcal{P}^c$ the functions $\tilde{F}^c_{\lambda/\mu}$ given by

$$\tilde{F}^c_{\lambda/\mu}(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle h_{a_t}(u) h_{a_{t-1}}(u) \cdots h_{a_1}(u) \cdot \mu, \lambda \right\rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$$

are the cylindric Schur functions of [McN, Pos]. They are chains $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r \subset \mu$ of cylindric partitions such that each λ^i/λ^{i-1} has at most one box in each column. Note that $\lambda \subset \mu$ means that μ lies to the southeast of λ when considered as lattice paths. If $w \in \tilde{S}_n$ is an affine permutation of minimal length satisfying $w \cdot \mu = \lambda$ then $\tilde{F}^c_{\lambda/\mu}(X) = \tilde{F}_w(X)$. The element w will necessarily be 321-avoiding. Conversely, any affine Stanley symmetric function labelled by a 321-avoiding permutation is equal to a cylindric Schur function.

7. Positivity

We conjecture that affine Schur functions generalise Schur functions for Stanley symmetric function positivity.

Conjecture 14. *The affine Stanley symmetric functions $\tilde{F}_w(X)$ expand positively in terms of the affine Schur functions $\tilde{F}_\lambda(X)$.*

This conjecture seems to be consistent with all the known behaviour of k-Schur functions and cylindric Schur functions. In fact, it has been conjectured that the multiplicative constants for k-Schur functions are non-negative, which by duality would imply that the skew affine Schur functions expand positively in terms of affine Schur functions. Similarly, the conjecture seems to be consistent with Postnikov's result [Pos] that "toric" Schur polynomials (in finitely many variables) expand positively into Schur polynomials. The fact that in infinitely many variables the cylindric Schur functions are nearly never Schur positive can probably be reconciled via affine Schur positivity. See also McNamara's work on cylindric Schur positivity [McN].

Since $s^{(k)}_\lambda(X) \in \Lambda_{(n)}$ we have an element $s^{(k)}_\lambda(\mathbf{u}) \in \mathcal{U}_n$ (as before $k = n-1$). The following conjecture is inspired by the paper of Fomin-Greene [FG]. J. Morse has communicated to the author that a similar conjecture was studied by L. Lapointe and herself.

Conjecture 15. *The "non-commutative" k-Schur function $s^{(k)}_\lambda(\mathbf{u})$ can be written as a non-negative sum of monomials in $u_0, \ldots, u_{n-1}$.*

Proposition 16. *Conjecture 15 implies Conjecture 14.*

Proof. We compute using the affine Cauchy kernel that

$$\tilde{F}_w(X) = \sum_{\lambda \in \mathcal{P}ar^n} \langle h_\lambda(\mathbf{u}) \cdot 1, w \rangle \, m_\lambda(X) = \left\langle \Omega^{(n)} \cdot 1, w \right\rangle = \sum_{\lambda \in \mathcal{P}ar^n} \left\langle s^{(k)}_\lambda(\mathbf{u}) \cdot 1, w \right\rangle \tilde{F}_\lambda(X).$$

Since u_i acts with non-negative coefficients, Conjecture 15 now implies that the coefficients $\left\langle s^{(k)}_\lambda(\mathbf{u}) \cdot 1, w \right\rangle$ are non-negative. □

8. Final comments

8.1. **The affine flag variety, quantum cohomology and fusion ring.** The connections with k-Schur functions and with cylindric Schur functions indicate that our definition of affine Stanley symmetric functions are indeed the correct definitions. Shimozono has conjectured that the multiplication of k-Schur functions calculate the homology multiplication of the affine Grassmannian. Multiplication of k-Schur functions is related to co-multiplication of affine Schur functions which are special cases of affine Stanley symmetric functions (for Grassmannian affine permutations). Thus it seems plausible that affine Stanley symmetric functions in general should be related to the affine flag variety. The direction towards *affine Schubert polynomials* seems to be the most fruitful one to take.

Note that our results show directly that k-Schur functions and cylindric Schur functions are related. This was already known if we combine Postnikov's work on cylindric Schur functions and Gromov-Witten invariants of the Grassmannian with Lapointe and Morse's work on k-Schur functions and the fusion ring (also known as the Verlinde algebra). Finally it is known that the fusion ring agrees with the quantum cohomology of the Grassmannian at $q = 1$. This suggests that there may be an interesting q-analogue of our theory.

8.2. **Affine stable Grothendieck polynomials.** Whereas Schubert polynomials are representatives for the cohomology of the flag variety, *Grothendieck polynomials* are representatives for the K-theory of the flag variety. In the same way that Stanley symmetric functions are stable Schubert polynomials, one can define *stable Grothendieck polynomials*. Our definition of affine Stanley symmetric functions naturally generalises to a definition of affine stable Grothendieck polynomials.

Let $\tilde{\mathcal{U}}_n$ be the algebra obtained from $\mathcal{U}_n$ by replacing the relation $u_i^2 = 0$ with $u_i^2 = 1$. Define $\tilde{h}_k(\mathbf{u}) \in \tilde{\mathcal{U}}_n$ for $k \in [0, n-1]$ with the same formula as for $h_k(\mathbf{u})$.

Definition 17. Let $w \in \tilde{S}_n$. The affine stable Grothendieck polynomial $\tilde{G}_w(X)$ is

$$\tilde{G}_w(X) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle \tilde{h}_{a_t}(u)\tilde{h}_{a_{t-1}}(u)\cdots\tilde{h}_{a_1}(u)\cdot 1, w \right\rangle x_1^{a_1}x_2^{a_2}\cdots x_t^{a_t},$$

where the sum is over compositions of $l(w)$ satisfying $a_i \in [0, n-1]$.

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