**TOTAL POSITIVITY AND THE AMPLITUHEDRON: EXERCISES**

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## Lecture 1: Totally Positive Spaces

1. Let $V \in \mathrm{Gr}(2,n)_{>0}$ be represented by a matrix with columns $v_1, v_2, \ldots, v_n \in \mathbb{R}^2$. In the lecture, we explained that the vectors $v_1, v_2, \ldots$ are arranged in counter-clockwise order, and all of them belong to a halfspace. We saw some degenerations of such a vector collection in the lecture. How many distinct combinatorial types of degenerations are there for $\mathrm{Gr}(2,4)_{>0}$?

2. Recall that $\tau \in \mathrm{End}(\mathbb{R}^n)$ is the symmetric matrix $\tau = S + S^T$ where $S([v_1, \ldots, v_n]) = [v_2, \ldots, v_n, (-1)^{k-1}v_1]$ and $S^T([v_1, \ldots, v_n]) = [(-1)^{k-1}v_n, v_1, \ldots, v_{n-1}]$. Verify that the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ for $\tau$ are given by

- if $k$ is even: $\lambda_1 = \lambda_2 = 2\cos(\pi/n), \lambda_3 = \lambda_4 = 2\cos(3\pi/n), \lambda_5 = \lambda_6 = 2\cos(5\pi/n), \ldots$
- if $k$ is odd: $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2\cos(2\pi/n), \lambda_4 = \lambda_5 = 2\cos(4\pi/n), \ldots$

3. Let $u_1, u_2, \ldots, u_n$ be an orthogonal basis of eigenvectors of $\tau$ with eigenvalues $\lambda_1, \lambda_2, \ldots$ respectively.
(a) Verify that $X_0 = \mathrm{span}(u_1, \ldots, u_n)$ is the unique cyclically invariant point in $\mathrm{Gr}(k,n)_{\geq 0}$.
(b) Show that the Plücker coordinates of $X_0$ are given by

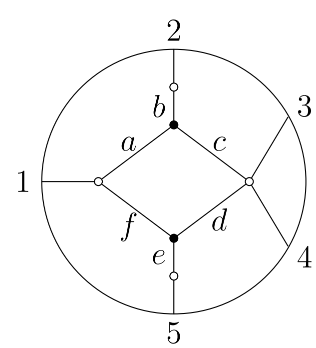
$$\Delta_I(X_0) = \prod_{i,j \in I, i<j} \sin\left(\frac{j-i}{n}\pi\right) > 0$$

for all $I \in \binom{[n]}{k}$.

4. Let $N$ be a planar bipartite graph embedded in the disk with $n$ boundary vertices and real positive edge weights. Assume that only white vertices are incident to edges from the boundary. An *almost perfect matching* $\Pi$ in $N$ is a collection of edges that is incident to each interior vertex exactly once. Let $\partial(\Pi)$ be the set of boundary vertices used by $\Pi$.
(a) Let $k$ be the cardinality of $\partial(\Pi)$. Show that $k$ does not depend on the almost perfect matching $\Pi$.
(b) Define the *boundary measurements* of $N$ as the generating function

$$\Delta_I(N) = \sum_{\substack{\Pi \\ \partial(\Pi)=I}} \mathrm{wt}(\Pi),$$

where $I \in \binom{[n]}{k}$ and $\mathrm{wt}(\Pi)$ is the product of the weights of the edges used in $\Pi$. Compute the boundary measurements for the graph $N$ pictured below (if an edge is not labeled, then its weight is 1). Verify that the boundary measurements define a point $X(N) \in \mathrm{Gr}(2,5)_{\geq 0}$.



(c) Can you prove that for any $N$ the boundary measurements (as long as they are not all 0) define a point in $\mathrm{Gr}(k,n)_{\geq 0}$?

5. Let $Y \in \mathrm{Gr}(k, k+m)$ be represented by a matrix with rows $y_1, \ldots, y_k$, and let $Z$ be a full rank linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$ with rows $Z_1, \ldots, Z_n$. Recall that given $i_1, \ldots, i_m \subset [n]$ we define the twistor coordinate

$$\langle i_1, \ldots, i_m \rangle = \langle Y Z_{i_1} \ldots Z_{i_m} \rangle$$

to be the determinant of the $(k+m) \times (k+m)$ matrix with rows $y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m}$. Given $C \in \mathrm{Gr}(k,n)$ with $Y = Z(C)$, verify that the following identity holds:

$$\langle i_1, \ldots, i_m \rangle = \sum_{I \in \binom{[n]}{k}} \Delta_I(C) \Delta_{I i_1 \ldots i_m}(Z)$$

6. Show that the twistor coordinates $\langle i_1, \ldots, i_m \rangle$ satisfy the Plücker relations of $\mathrm{Gr}(m,n)$. That is, show that

$$\langle i_1, \ldots, i_m \rangle \langle j_1, \ldots, j_m \rangle = \sum \langle i_1', \ldots, i_m' \rangle \langle j_1', \ldots, j_m' \rangle$$

where the sum on the right hand side is over all pairs obtained by interchanging a fixed set of $r$ of the subscripts $j_1, \ldots, j_,$ with $r$ of the subscripts $i_1, \ldots, i_m$, maintaining the order in each.

7. Let $Z$ be a linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$. Consider the map $\mathcal{A}_{n,k,m}(Z) \to \mathrm{Gr}(m,n)$ given by sending $Y \in \mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}(k, k+m)$ to the point in $\mathrm{Gr}(m,n)$ defined by the Plücker coordinates $\Delta_I = \langle i_1, \ldots, i_m \rangle$ for $I = \{i_1 < i_2 < \cdots < i_n\} \in \binom{[n]}{k}$. Show that this map is injective if $Z$ is full rank.

8. Verify the following lemma from the lecture for the case where $k = 2$ and $n = 4$:

**Lemma 1.** *For* $X \in \mathrm{Gr}(k,n)_{\geq 0}$ *and* $t > 0$, $\exp(t\tau)X \in \mathrm{Gr}(k,n)_{>0}$

Recall the following definitions from the lecture:

**Definition 1.** *A $(k,n)$-bounded affine permutation is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that*

*(1) $f(i+n) = f(i) + n$ for all $i$*
*(2) $i \leq f(i) \leq i+n$*
*(3) $\sum_{i=1}^{n} (f(i) - i) = kn$*

Given $X \in \mathrm{Gr}(k,n)$ we can associate to it a $(k,n)$-bounded affine permutation $f_X$ as follows. Suppose $X$ is represented by a matrix with columns $v_1, \ldots, v_n$, and set $v_{i+n} = v_i$ for all $i$. Define $f_X : \mathbb{Z} \to \mathbb{Z}$ by $f_X(i) = \min\{j \geq i \mid v_i \in \mathrm{span}(v_{i+1}, \ldots v_j)\}$.

**Definition 2.** *A matroid $\mathcal{M}$ is called a positroid if there is $X \in \mathrm{Gr}(k,n)_{\geq 0}$ such that $\mathcal{M} = \mathcal{M}_X := \{I \in \binom{[n]}{k} \mid \Delta_I(X) \neq 0\}$.*

**Definition 3.** *A $(k,n)$-Grassmann necklace is a collection $\mathcal{I} = (I_1, \ldots, I_n)$ of $k$-element subsets $I_a$ such that for each $a \in [n]$ the following conditions hold:*

*(1) $I_{a+1} = I_a$ if $a \notin I_a$*
*(2) $I_{a+1} = I_a - \{a\} \cup \{a'\}$ for some $a'$ if $a \in I_a$*

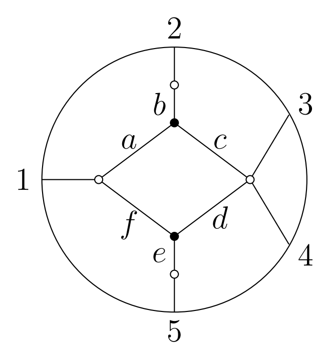
Given a rank $k$ matroid $\mathcal{M}$ on $[n]$ there is an associated Grassmann necklace $\mathcal{I}(\mathcal{M})$ of $\mathcal{M}$ is $(I_1, \ldots, I_n)$ where $I_a$ is the lexicographically minimal base of $\mathcal{M}$ with respect to the cyclically shifted order $\leq_a$ where $a$ is minimal.

1. Let $X \in \mathrm{Gr}(k,n)$. Verify that the map $f_X$ defined after Definition 1 above is a $(k,n)$-bounded affine permutation.

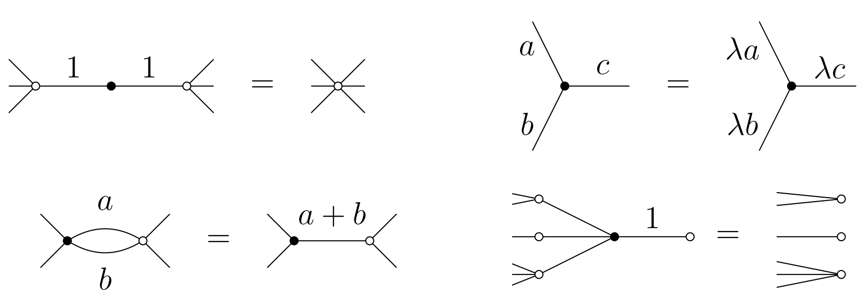
2. Consider the bounded affine permutation $f = [2,5,6,4,8]$. Compute the corresponding positroid and Grassmann necklace.

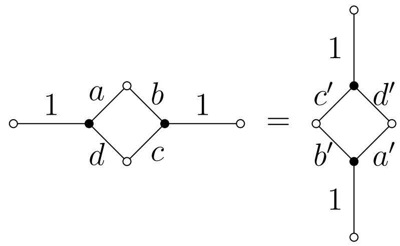
3. Describe explicitly the compatible bijections between positroids, Grassmann necklaces, and bounded affine permutations when $k = 1$. Compare with the faces of a simplex.

4. Recall the below graph $N$ from the previous set of exercises. Compute the bounded affine permutation, Grassmann necklace, and positroid of the point $X(N)$.




5. Verify that each of the moves pictured below on planar bipartite graphs $N$ preserves the corresponding point $X(N)$ in the Grassmannian.






where $\lambda \in \mathbb{R}_{>0}$, $a' = \frac{a}{\Delta}$ for $\Delta = ac + bd$, and $b', c'$, and $d'$ are defined similarly.

6. Let $\mathcal{M}$ be a positroid of rank $k$ on ground set $[n]$. The dual matroid $\mathcal{M}^* = \{[n] \setminus I \mid I \in \mathcal{M}\}$ is also a positroid. What is the Grassmann necklace of $\mathcal{M}^*$?

7. Let $f$ be a $(k, n)$-bounded affine permutation. Let $\mathcal{M}$ be the matroid corresponding to $f$, i.e. the matroid $\mathcal{M}_X$ of any $X \in \mathrm{Gr}(k, n)_{\geq 0}$ with $f_X = f$. Show that

a) if $f(i) = i$, then $i$ is a loop in $\mathcal{M}$, i.e. $i$ is in no base of $\mathcal{M}$

b) if $f(i) = i + n$, then $i$ is a coloop in $\mathcal{M}$, i.e. $i$ is in all bases of $\mathcal{M}$.

8. There is a partial order $\preceq$ on positroid cells given by $\Pi_{\mathcal{M},>0} \preceq \Pi_{\mathcal{M}',>0}$ if and only if $\Pi_{\mathcal{M},>0} \subseteq \overline{\Pi_{\mathcal{M}',>0}}$. Describe the cover relations on the positroid cells of $\mathrm{Gr}(2, 4)_{\geq 0}$ given by this partial order.

10. Given a plabic graph $G$ with edge set $E$ and vertex set $V$, we can parametrize $\Pi_{f_G,>0}$ by placing weights on $|E| - |V|$ of the edges in $G$. On which edges should the weights be placed?

# LECTURE 3: THE $m = 2$ AMPLITUHEDRON

Let $Q_{n,2}$ be the poset of rank 2 positroids ordered by inclusion. Recall from the lecture the upper order ideal $P_{n,k} \subset Q_{n,2}$, which is generated by the $\binom{n}{k}$ positroids $\mathcal{N}(L)$, where $|L| = k$ is the set of loops and $\mathcal{N}(L)|_{[n]-L}$ is the uniform matroid.

1. Take $k = 1, m = 2, n = 5$. The amplituhedron $A_{5,1,2}$ is a pentagon. Investigate the face poset and triangulations of the pentagon in the language of positroids.
   (a) What are the $(1, 5)$ bounded-affine permutations $f$? Classify the images $Z(\Pi_{f,\geq 0})$. When is it a triangle, square? When is it on the boundary of the pentagon?
   (b) What are the positroids of rank 1 on $[5]$? Compute the twistor map on these matroids.
   (c) Compare the poset $P_{5,1}$ with the face poset of the pentagon.

2. Take $k = 2, m = 2, n = 5$. The amplituhedron $A_{5,2,2}$ is a 4-dim subspace of $\mathrm{Gr}(2, 4)$.
   (a) What are the $(2, 5)$ bounded-affine permutations $f$? Investigate the dimensions of various $Z(\Pi_{f,\geq 0})$, and whether they lie on the boundary of $A_{5,2,2}$. (Hint: look at which twistor coordinates $\langle ab \rangle$ vanish on $\Pi_f$.)
   (b) Try to find a triangulation of $A_{5,2,2}$. Can you prove it?
   (c) Draw the facet poset $P_{5,2}$ of the amplituhedron. What are the face numbers of $A_{5,2,2}$?

3. Choose a triangulation of the pentagon and check the statement of *parity duality*: if $f_1, f_2, \ldots, f_r$ form a triangulation of $A_{n,k,m}$ (with $m$ even) then $g_1, \ldots, g_r$ form a triangulation of $A_{n,n-k-m,m}$, where

$$g_i = (f_i - k)^{-1} + (n - k - m).$$

Here, $f - k : \mathbb{Z} \to \mathbb{Z}$ is the bijection given by $(f - k)(a) = f(a) - k$, and $f^{-1}$ is the inverse bijection.

4. Take the real Grassmannian $\mathrm{Gr}(2, 4)$ and remove from it the four positroid divisors $\{\Delta_{i,i+1} = 0\}$. Compute the number of connected components of the resulting space. (Hint: find a free action of the torus $(\mathbb{R}^\times)^2$ on this space to reduce to lower-dimensional space.)

5. Recall that the $m$-twistor of a matroid $\mathcal{M}$ is given by

$$\mathcal{M}^{\downarrow m} = \{I \in \binom{[n]}{m} \mid I \cap J = \emptyset \text{ for some } J \in \mathcal{M}\}$$

where we think of matroids as collections of bases. For a rank 2 positroid $\mathcal{N}$, we let $\mathcal{N}^{\uparrow k}$ be the largest matroid such that $\mathrm{env}((\mathcal{N}^{\uparrow k})^{\downarrow 2}) = \mathcal{N}$. Here, for a matroid $\mathcal{M}$, we denote by $\mathrm{env}(\mathcal{M})$ the *positroid envelope*: the smallest positroid containing $\mathcal{M}$.

(a) Let $\mathcal{N}$ be the rank 2 positroid on $[8]$ with a loop 5 and the rank conditions $\mathrm{rank}(2, 3) = 1$ and $\mathrm{rank}(4, 5, 6, 7) = 1$. What is the positroid $\mathcal{N}^{\uparrow 4}$? Write down a Lukowski matrix for $\mathcal{N}^{\uparrow 4}$ and verify that it has the correct positroid.

(b) In general, when is $\mathcal{N}^{\uparrow k}$ well-defined? (Here, $\mathcal{N}$ is an arbitrary matroid of arbitrary rank.)