Dimers and Webs.

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arXiv : 1404.3317

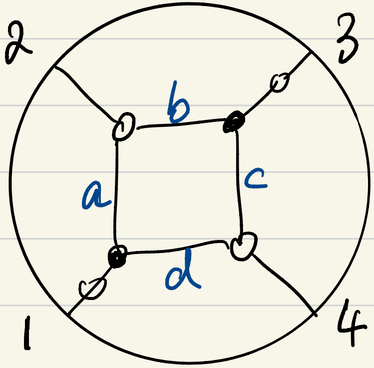
arXiv : 1506.00603

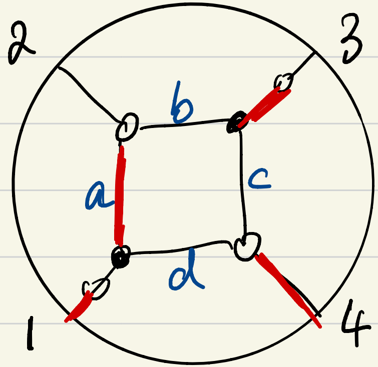
} Lecture 1

arXiv : 1705.09424 (j. C. Fraser and I. Le) Lecture 2

Review: Dimers → Grassmannian

$N =$





$\partial(\pi) = \{1,4\}$

$wt(\pi) = a$

planar bipartite graph

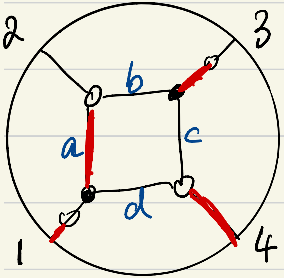
boundary vertices are black and degree one

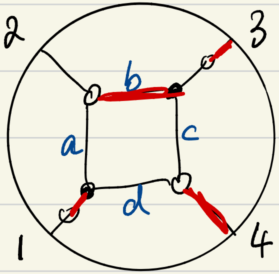
π = almost perfect matching (uses all interior vertices once.)

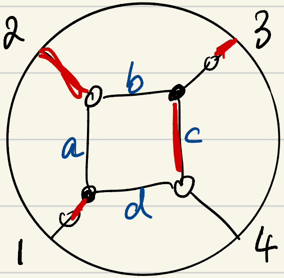
$$\Delta_I(N) = \sum_{\pi : \partial(\pi) = I} wt(\pi)$$

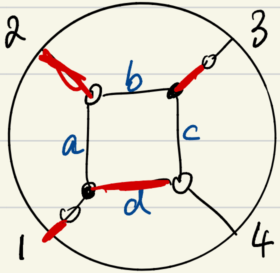
$|I| = k :=$ # interior white − # interior black

Example of dimer generating function:

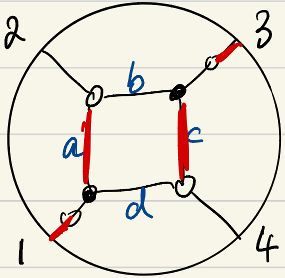


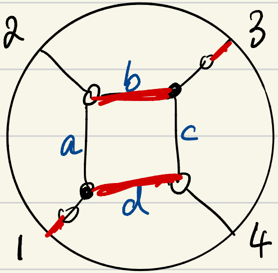


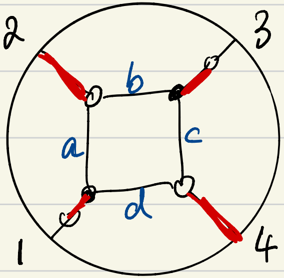




wt	a	b	c	d
∂	14	34	23	12







$$\Delta_{13}(N) = ac+bd$$

ac	bd	1
13	13	24

Theorem 1 [Kuo
Postnikov, Talaska, Postnikov - Speyer Williams
L.]

$(\Delta_I(N))_{I \in \binom{[n]}{k}}$ defines a point in $Gr(k,n)$

ie. satisfies the Plücker relations

$$\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1, i_2, \dots i_{k-1}, j_s} \Delta_{j_1, \dots \hat{j_s} \dots j_{k+1}} = 0$$

e.g. $k=2$ $i_1 = 1$ $j_1, j_2, j_3 = 2, 3, 4$.

$$-\Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} - \Delta_{14} \Delta_{23} = 0.$$

$$-d.b + (ac+bd) \cdot 1 - a.c = 0$$

Proof using "webs".

Definition: A subgraph $\Sigma \subset N$ is 2-weblike if it is a union of connected components:

(a) paths between boundary vertices
(b) interior cycles
(c) single edges (called "doubled edge")

such that every interior vertex is used.

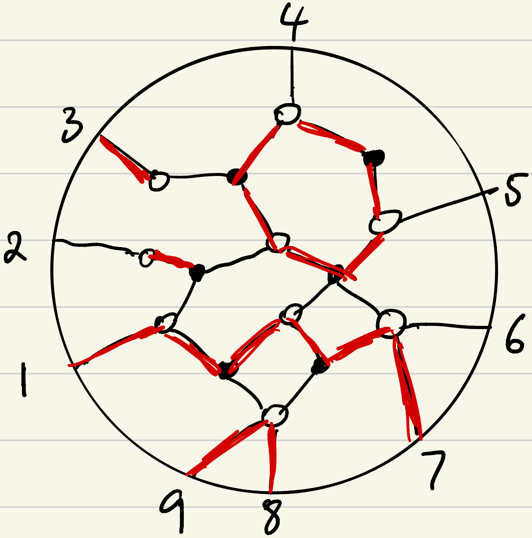
$$F_{\tau,T}(N) := \sum_{\Sigma} wt(\Sigma)$$

Temperley-Lieb immanant.

[L., cf. Rhoades-Skandera
cf. Kenyon-Wilson]

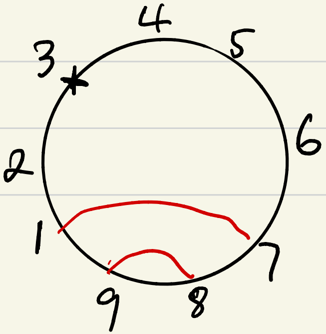
τ = partial non crossing matching
T = boundary vertices used in doubled edges

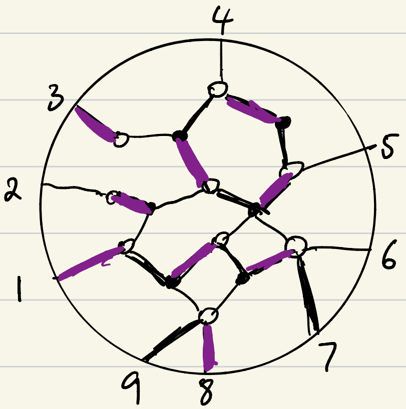
$$wt(\Sigma) = \prod_{\substack{e \in \text{path} \\ \text{or} \\ \text{cycle}}} wt(e) \cdot \prod_{\substack{e' \in \\ \text{doubled} \\ \text{edge}}} wt(e')^2 \cdot 2^{\#\text{cycle}}$$



Σ

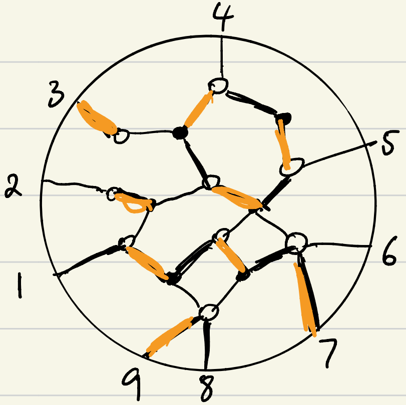
$\tau, T =$





$I =$ 138

Double dimer



$J = 379$

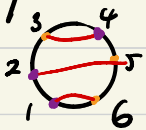
Definition $I,J\in\binom{[n]}{k}$ is compatible with (τ,T) if:

- $T= I\cap J$
- each chord of τ matches $i\in I-J$ with $j\in J-I$

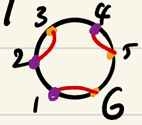
Theorem 2 [L. cf. Kenyon-Wilson, Rhoades-Skandera, ...]

$$\Delta_I(N)\Delta_J(N)=\sum_{\substack{(\tau,T)\\ \text{comp. with } (I,J)}} F_{\tau,T}(N)$$

e.g. $\Delta_{124}\Delta_{356} = F$



$+ \; F$



$R(k,n):=\mathbb{C}[Gr(k,n)]$

$R(k,n)=\bigoplus_{d\geq 0} R(k,n)_d$

$R(k,n)_1=\bigoplus \mathbb{C}\cdot\Delta_I$

$R(k,n)_2=\bigoplus \mathbb{C}\cdot F_{\tau,T}$

- $F_{\tau,T}$ is a well-defined quadratic function on $\hat{Gr}(k,n)$
- forms a cyclically invariant basis
- coincides with "dual canonical basis", better than "standard monomial basis"

$\Delta_{S_1}\Delta_{S_2}$ $\quad \boxed{S_1|S_2}$ semi standard

- Theorem 2 ⇒ Theorem 1

Expand $\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1,i_2,\dots i_{k-1},j_s} \Delta_{j_1,\dots \hat{j}_s \dots j_{k+1}}$ in terms of $F_{\tau,T}$

and do a sign-reversing involution.

- $\Delta_I \Delta_J \leq \Delta_{\min(I,J)} \Delta_{\max(I,J)} \leq \Delta_{\mathrm{sort}_1(I,J)} \Delta_{\mathrm{sort}_2(I,J)}$

e.g. $\Delta_{13567} \Delta_{23489} \leq \Delta_{13467} \Delta_{23589} \leq \Delta_{13468} \Delta_{23579}$

sort $= 123345678 9$

[cf. Rhoades-Skandera
Farber-Postnikov
L.-Postnikov-Pylyavskyy]

- $\mathcal{M}(N) := \{ I \in \binom{[n]}{k} \mid \Delta_I(N) \neq 0 \}$ matroid of N.

  $I, J \in \mathcal{M} \Rightarrow \mathrm{sort}_1, \mathrm{sort}_2 \in \mathcal{M}$ [L.-Postnikov]

  "sort-closed matroid"

- $\mathcal{M}^{(2)}(N) := \{ \tau, T \mid F_{\tau,T}(N) \neq 0 \}$ "quadratic matroid"

- any positive quadratic function, homogeneous with respect to torus action, is a positive linear combination of $F_{\tau,T}$

- applications to dimers?

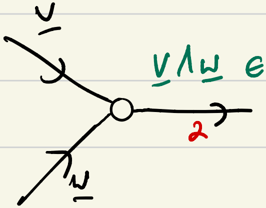
Webs

Based on joint work with C. Fraser and I. Le

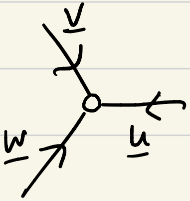
$$k\left\{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}\right.$$

$$\mathbb{C}[Gr(k,n)] = R(k,n) = SL_k \text{ invariant functions on } \underbrace{\mathbb{C}^k \times \cdots \times \mathbb{C}^k}_{n} = \mathbb{C}[\Delta_I]/\text{relations}$$

$k=3$

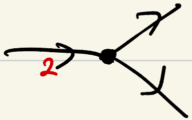


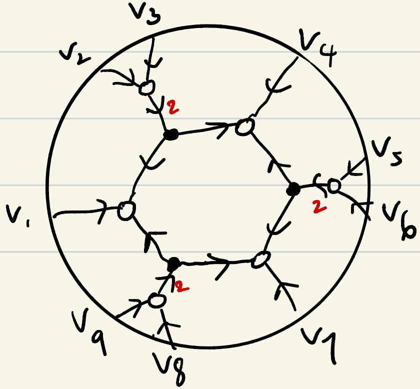
$\underline{v}\wedge\underline{w} \in \Lambda^2(\mathbb{C}^3)$



$\det(\underline{v}|\underline{w}|\underline{u})$

$\Lambda^2\mathbb{C}^3 \hookrightarrow \mathbb{C}^3 \otimes \mathbb{C}^3$





$\in R(3,9)_3$

$v_i \in \mathbb{C}^3$

[Reshetikhin–Turaev
Kuperberg]

Clusters and webs

[Fomin–Pylyavskyy]

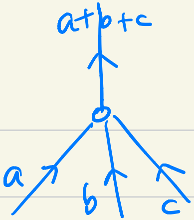
Cluster structure of $\mathbb{C}[Gr(3,n)]$ (+ more) can be described in terms of web invariants:

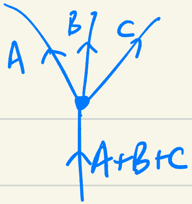
- initial cluster, quivers, coefficients

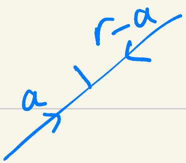
Conjecture [Fraser Gr(3,9)]

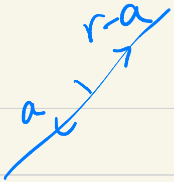
- cluster variables are indecomp. web invariants
- " monomials " tensor invariants
- compatibility

SL(r)-webs

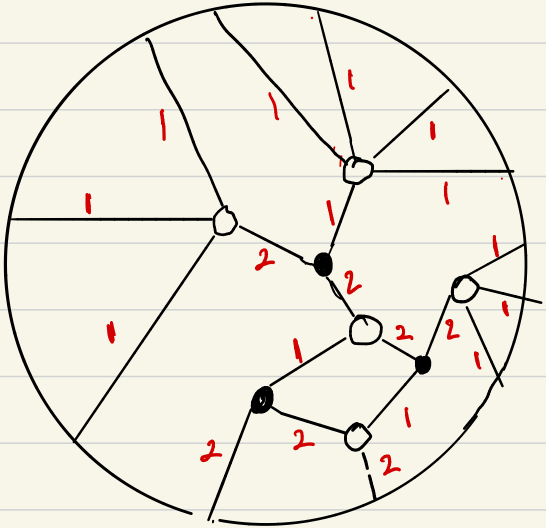






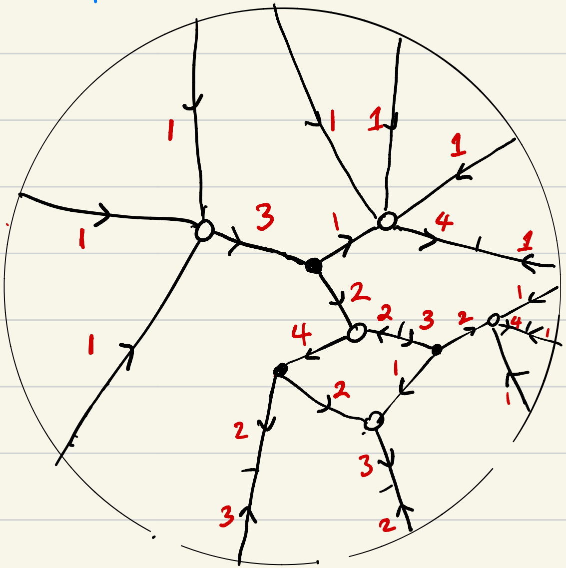


r=5

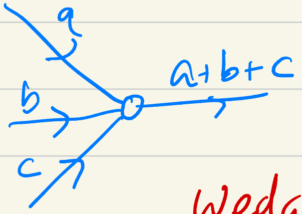


untagged SL(5)-web

$\lambda = (1,1,1,\dots,1,$
$2,3)$



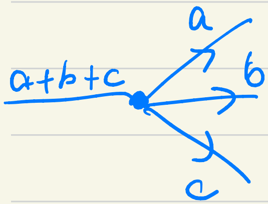
tagged SL(5)-web



wedge

$$\Lambda^a(\mathbb{C}^r)\otimes\Lambda^b(\mathbb{C}^r)\otimes\Lambda^c(\mathbb{C}^r)\to\Lambda^{a+b+c}(\mathbb{C}^r)$$

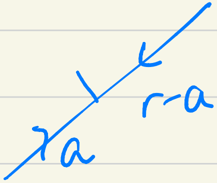
$$x_1\otimes x_2\otimes x_3\longmapsto x_1\wedge x_2\wedge x_3$$



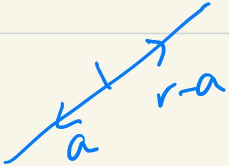
shuffle

$$\Lambda^{a+b+c}(\mathbb{C}^r)\to\Lambda^a(\mathbb{C}^r)\otimes\Lambda^b(\mathbb{C}^r)\otimes\Lambda^c(\mathbb{C}^r)$$

$$x_1\wedge\dots\wedge x_{a+b+c}\longrightarrow\sum\pm(x_{i_1}\wedge\dots\wedge x_{i_a})\otimes(\qquad)\otimes(\qquad$$



$$\Lambda^a(\mathbb{C}^r)\otimes\Lambda^{r-a}(\mathbb{C}^r)\longrightarrow\Lambda^r(\mathbb{C}^r)\cong\mathbb{C}$$



$$\Lambda^r(\mathbb{C}^r)\longrightarrow\Lambda^a(\mathbb{C}^r)\otimes\Lambda^{r-a}(\mathbb{C}^r)$$

Lemma. Any tagging $\hat{W}$ of a web W produces the same function up to sign.

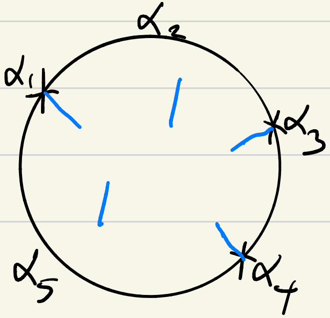
$$W_\lambda(r) = \mathrm{Hom}_{SL(r)}\left(\bigotimes_{i=1}^{n} \Lambda^{\lambda_i} \mathbb{C}^r, \mathbb{C}\right) \qquad \lambda = (\lambda_1, \ldots, \lambda_n)$$

$$W(r,n) := \mathrm{Hom}_{SL(r)}\left(\bigotimes_{i=1}^{n} \mathbb{C}^r, \mathbb{C}\right)$$

are spanned by appropriate $\hat{W}$.

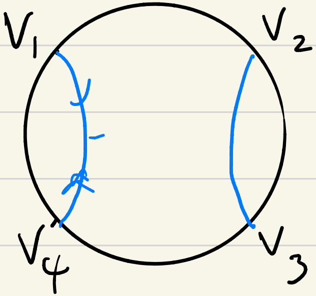
Ex. $r=1$

$\alpha_i \in \mathbb{C}$



$$\hat{W} = \pm \alpha_1 \alpha_3 \alpha_4$$

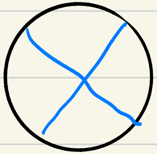
$r=2$ $\qquad$ $v_i \in \mathbb{C}^2$



$$= \pm \det(v_1|v_4)\det(v_2|v_3)$$

basis of $W(2,2n)$ = noncrossing matchings on $[2n]$

[Rumer-Teller-Weyl]



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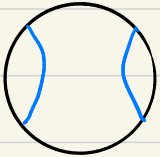
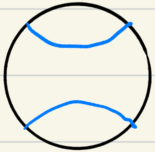
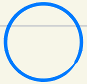
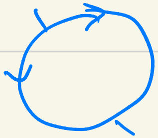


+



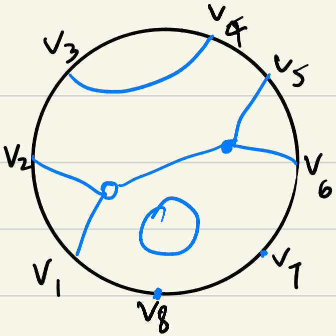


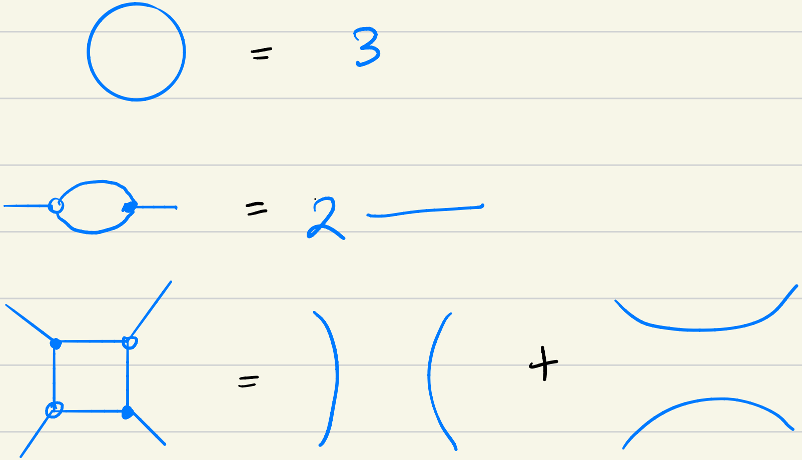
$= \pm 2 \qquad = \operatorname{trace}(\mathrm{Id}: \mathbb{C}^2 \to \mathbb{C}^2)$



$r=3$

e.g.





[Kuperberg] basis = irreducible / elliptic webs

ie. no contractible loops, no square faces

Definition An r-weblike subgraph $W \subset N$ is a subgraph using all vertices of N, with each edge labeled by a multiplicity $m(e)$, such that

$$\sum_{e \text{ incident to } v} m(e) = r \qquad \text{for interior } v$$

$$\mathrm{wt}(W) = \prod_e \mathrm{wt}(e)^{m(e)}$$

$\lambda(W)$ = boundary multiplicities.

Each such W gives an untagged $SL(r)$-web and a tensor invariant $\underline{W}$ $\left(= \pm \hat{W} \text{ for some tagging}\right)$

$$\boxed{\mathrm{Web}_r(N;\lambda) := \sum_{\substack{\mathrm{WCG} \\ \lambda(W)=\lambda}} \mathrm{wt}(W)\,\underline{W} \in \mathcal{W}_\lambda(r)}$$

- $r=1$ $\quad \lambda \longleftrightarrow I \quad \mathrm{Web}_1(N;\lambda) = \Delta_I(N)$

Theorem 1 If $\Delta_I(N) = \Delta_I(N')$ for all I, then
[Fraser, L., Le]

$$\mathrm{Web}_r(N;\lambda) = \mathrm{Web}_r(N';\lambda) \in \mathcal{W}_\lambda(r)$$

for all r, λ.

$\mathbb{C}[Gr(k,n)]_\lambda$ = be the λ-weight component.

e.g. $\Delta_{124}\Delta_{246}$ has weight $(1,2,0,2,0,1)$

Theorem 2 Imm: $W_\lambda(U)^* \longrightarrow \mathbb{C}[Gr(k,n)]_\lambda$

$$\varphi \longmapsto \left(N \longmapsto \varphi(Web_r(N;\lambda))\right)$$

is an isomorphism.

Theorem 3 $\Delta_{I_1}(N)\Delta_{I_2}(N)\dots\Delta_{I_r}(N) = sign(S)\, Web_r(N;\lambda)\big|_{e_S}$

where

$S=(S(1),\dots,S(n))$ $S(j)\subset\binom{[r]}{\lambda_j}$

$e_S := e_{S(1)}\otimes\dots\otimes e_{S(n)} \in \Lambda^{\lambda_1}(\mathbb{C}^r)\otimes\dots\otimes\Lambda^{\lambda_n}(\mathbb{C}^r)$

$(e_{124} = e_1\wedge e_2\wedge e_4)$

I, S related by

$I_i = \{j\in[n] : i\in S(j)\}$

ie. transpose as 01-matrices.

Remarks

- $r=2$ $\qquad W_\lambda(2)$ has noncrossing matching basis:

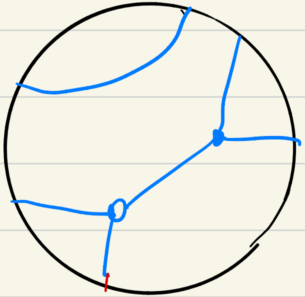
e.g.

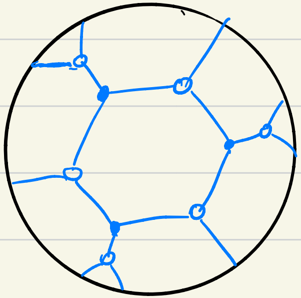
$\in W_{(1,1,1,1,1,1)}(2) = W(2,6) \xleftrightarrow[\text{dual}]{} \mathbb{C}[Gr(3,6)]_{(1,1,1,\ldots)}$

$\mathrm{Imm}(\text{dual basis element}) = F_{\tau,T}$.

- $r=3$

[Kuperberg] $W_\lambda(3)$ has basis of irreducible/elliptic webs.





no square faces!

$\text{Imm}(\text{dual basis element}) =$ "web immanant" [L. cf. Pylyavskyy]

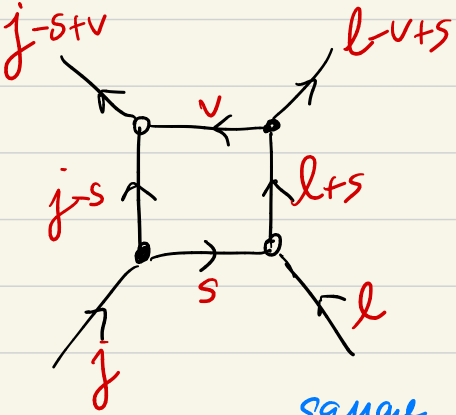
- For $r>3$, no generally agreed upon web basis is known. There is a dual canonical basis.

- generators for relations b/w webs are known

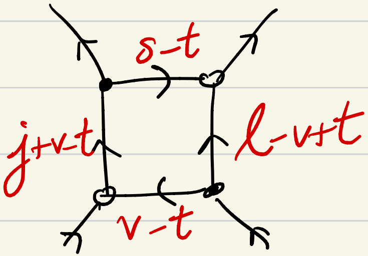
[Kuperberg, . . . , Cautis – Kamnitzer – Morrison]

Theorem Under $\mathrm{Web}_r(N;\lambda) = \mathrm{Web}_r(N';\lambda)$

local moves for planar bipartite graphs ⇒ web relations



$$= \sum_{t} \binom{j-l+v-s}{t}$$



square move for tagged webs

Further directions

- Quantization?
- Positivity of $\underline{W}$ as function on $Gr(r,n)$?
- Applications to dimers?
- Relation Imm to cluster structure?