NOTES ON POSITIVE GEOMETRY 01

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1. Polytopes are positive geometries

1.1. **Definitions.** We will be working with convex projective polytopes $P \subset \mathbb{P}^m(\mathbb{R}) = \{[X_0 : X_1 : \cdots : X_m]\}$. We can define it by convex hull, where $Z_i \in \mathbb{R}^{m+1}$:

$$P = \mathrm{Conv}(Z_1, \ldots, Z_n) := \left\{ \sum_{i=1}^{n} c_i Z_i \in \mathbb{P}^m \mid c_i \geq 0, i = 1, \ldots, n \right\}.$$

We typically make the assumption that $Z_1, \ldots, Z_n$ are vertices and that $\sum c_i Z_i = 0$ if and only if $c_i = 0$ for all i.

To think about a projective polytope in affine spaces, we have two very natural ways. And you are very encouraged to think in affine spaces so that I can draw pictures. The first is the affine cone

$$\mathrm{Cone}(P) = \left\{ \sum_{i=1}^{n} c_i Z_i \in \mathbb{R}^{m+1} \mid c_i \geq 0, i = 1, \ldots, n \right\}.$$

And the second is the polytope in chart $X_0 = 1$, where $Z = (1, Z')$:

$$P = \left\{ \sum_{i=1}^{n} c_i Z'_i \in \mathbb{R}^m \mid c_i \geq 0, \sum c_i = 1 \right\}.$$

1.2. **Standard simplices are positive geometries.** The *standard simplex* $\Delta^m := \mathbb{P}^m_{\geq 0}$ is the convex hull of coordinate vectors, which is the set of points in $\mathbb{P}^m(\mathbb{R})$ representable by nonnegative coordinates. We claim that $(\mathbb{P}^m, \Delta^m)$ is a positive geometry whose canonical form is given by

$$\Omega(\Delta^m) = \prod_{i=1}^{m} \frac{dx_i}{x_i} = \prod_{i=1}^{m} d \log x_i,$$

where we are on chart $X_0 = 1$ and $x_i = X_i / X_0$.

I know little about differential forms. To compute the residue, the following tool is useful:

$$\mathrm{Res}_x \left(w \wedge \frac{dx}{x} \right) = w\big|_{x=0}.$$

Let's check that the form works. The base case is $m = 1$ and $\Omega(\Delta^m) = dx_1/x_1$ so its residue at $x_1 = 0$, which is the point $e_0 = [1 : 0]$, is 1. To compute the residue at $e_1 = [0 : 1]$, we need to

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change a chart. Recall $x_1 = X_1/X_0$. Let $y_0 = X_0/X_1 = 1/x_1$ so that at chart $X_1 = 1$,

$$\Omega(\Delta^m) = \frac{d(1/y_0)}{1/y_0} = -\frac{dy_0}{y_0}$$

and this means that the residue at e_1 is -1.

For general m, we check inductively. There is really nothing to do for the residue at facets $X_i = 0$ for $i = 1, \ldots, m$, but let us note that the sign alternates. To check the residue at the facet $X_0 = 0$, let's choose the chart $X_1 = 1$ and let $y_i = X_i/X_1$ for $i \neq 1$. A set of local parameters is $y_0, y_2, \ldots, y_m$. Then $x_1 = 1/y_0$ and $x_i = y_i/y_0$ for $i \geq 2$. And

$$\begin{aligned}\Omega(\Delta^m) =& \frac{d(1/y_0)}{1/y_0} \wedge \frac{d(y_2/y_0)}{y_2/y_0} \wedge \cdots \wedge \frac{d(y_m/y_0)}{y_m/y_0} \\ =& -\frac{dy_0}{y_0} \wedge \frac{dy_2/y_0}{y_2/y_0} \wedge \cdots \wedge \frac{dy_m/y_0}{y_m/y_0} \\ =& -\frac{dy_0}{y_0}\frac{dy_2}{y_2}\cdots\frac{dy_m}{y_m}.\end{aligned}$$

For those of you who know nothing about differential forms like me, we have $dx \wedge dx = 0$ so in the second step above, we only need to take the derivative with respect to y_i, $i \geq 0$, in $d(y_i/y_0)$.

There is a gauge-invariant way of writing this form. We have

$$\begin{aligned}\Omega(\Delta^m) =& \frac{dx_1}{x_1}\cdots\frac{dx_m}{x_m} = \frac{d(X_1/X_0)}{X_1/X_0}\cdots\frac{d(X_m/X_0)}{X_m/X_0} \\ =& \left(\frac{dX_1}{X_1} - \frac{dX_0}{X_0}\right)\cdots\left(\frac{dX_m}{X_m} - \frac{dX_0}{X_0}\right) \\ =& \sum_{i=0}^{m}(-1)^i\frac{dX_0}{X_0} \wedge \cdots \wedge \widehat{\frac{dX_i}{X_i}} \wedge \cdots \wedge \frac{dX_m}{X_m}\end{aligned}$$

which is denoted as $\frac{1}{m!}\langle X d^m X\rangle/(X_0 \cdots X_m)$ in the main reference [1].

1.3. **Simplices are positive geometries.** A *projective simplex* is cut out by exactly $m+1$ linear inequalities. To think about linear inequalities in projective spaces, we can first solve them in $\mathbb{R}^{m+1}$ and then consider the image via the rational map $\mathbb{R}^{m+1} \dashrightarrow \mathbb{P}^m$. Clearly, given any projective simplex Δ', there exists a unique element $g \in \mathrm{PGL}_m$ that maps Δ to Δ' (we can think of Δ and Δ' as $(m+1) \times (m+1)$ matrices via their facets) and extends to an ismorphism on $\mathbb{P}^m$. We can then push forward or pull back our canonical form on the standard simplex in a naive way according to this linear isomorphism g. Let's do a few examples for application in affine spaces.

Example 1.1. Recall $\Omega(\Delta^1) = \frac{dX_1}{X_1} - \frac{dX_0}{X_0}$. Let $\Delta' \in \mathbb{P}^1$ be the segment from $[1 : a]$ to $[1 : b]$, which is bounded by the facets $X_0 = aY_0 - Y_1$ and $X_1 = bY_0 - Y_1$, where Y_0, Y_1 are the homogeneous coordinates for Δ'. Then, at chart $Y_0 = 1$,

$$\begin{aligned}\Omega(\Delta') =& \frac{d(bY_0 - Y_1)}{bY_0 - Y_1} - \frac{d(aY_0 - Y_1)}{aY_0 - Y_1} \\ =& \frac{dy_1}{y_1 - b} - \frac{dy_1}{y_1 - a}.\end{aligned}$$

Example 1.2. Recall

$$\Omega(\Delta^2) = \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} - \frac{dX_0}{X_0} \wedge \frac{dX_2}{X_2} + \frac{dX_0}{X_0} \wedge \frac{dX_1}{X_1}.$$

Consider the triangle with vertices $[1 : a_0 : b_0], [1 : a_1 : b_1], [1 : a_2 : b_2]$ so that our map g is given by

$$\begin{aligned} X_2 \mapsto & (a_0 b_1 - a_1 b_0) Y_0 + (b_0 - b_1) Y_1 + (a_1 - a_0) Y_2, \\ X_1 \mapsto & (a_0 b_2 - a_2 b_0) Y_0 + (b_0 - b_2) Y_1 + (a_2 - a_0) Y_2, \\ X_0 \mapsto & (a_1 b_2 - a_2 b_1) Y_0 + (b_1 - b_2) Y_1 + (a_2 - a_1) Y_2. \end{aligned}$$

Okay this is too complicated but one can check.

1.4. **Polytopes are positive geometries.** Let's review some definitions of triangulations (Section 3 of [1]).

Definition 1.3. We say that $X_{i,\geq 0}$ triangulates $X_{\geq 0}$ if

(1) each $X_{i,>0}$ is contained in $X_{>0}$ and the orientation agree;
(2) the interiors $X_{i,>0}$ of $X_{i,\geq 0}$ are mutually disjoint;
(3) $\cup X_{i,\geq 0} = X_{\geq 0}$.

A closedly related concept more convenient for reasoning is that of *signed triangulations*.

Definition 1.4. We say that $X_{i,\geq 0}$ interior triangulates the empty set if for every point $x \in \cup_i X_{i,\geq 0}$ that does not lie in any boundary components of $X_{i,\geq 0}$, we have

$$\begin{aligned} & \#\{i \mid x \in X_{i,\geq 0} \text{ and } X_{i,>0} \text{ is positively oriented at } x\} \\ = & \#\{i \mid x \in X_{i,\geq 0} \text{ and } X_{i,>0} \text{ is negatively oriented at } x\}. \end{aligned}$$

If $\{X_{1,\geq 0}, \ldots, X_{t,\geq 0}\}$ interior triangulates the empty set, then we also say that $\{X_{2,\geq 0}, \ldots, X_{t,\geq 0}\}$ interior triangulates $X^-_{1,\geq 0}$. For example, if simplices $\{T_i\}$ triangulates our polytope P, then $\{T_i\} \cup \{P^-\}$ interior triangulates the empty set.

Proposition 1.5. *If $\{X_{i,\geq 0}\}$ interior triangulates the empty set, then $\sum \Omega(X_{i,\geq 0}) = 0$.*

In terms of polytopes, we translate Proposition 1.5 into the following.

Corollary 1.6. *If $\{T_i\}$ subdivides the polytope P, then $\Omega(P) = \sum \Omega(T_i)$.*

The proof of Proposition 1.5 relies on showing that $\mathrm{Res}_C \Omega = 0$ where $\Omega := \sum \Omega(X_{i,\geq 0})$, and C is any irreducible subvariety of X of codimension 1 and then concluding via induction that $\Omega = 0$. The analysis of local behaviour is essentially trivial and the notion of *boundary triangulation* is introduced in the process. Readers are refered to Appendix B of [1]. We sketch a proof of Corollary 1.6 in the same flavor.

Proof of Corollary 1.6. Let $\Omega = \sum \Omega(T_i)$. Let F be (the affine span of) a boundary component of P, or in other words, F is a facet of P. Then

$$\mathrm{Res}_F(\Omega) = \sum \mathrm{Res}_F \Omega(T_i) = \sum_{F \text{ is a facet of } T_i} \mathrm{Res}_F \Omega(T_i) = \sum_{F \text{ is a facet of } T_i} \Omega(C_i)$$

since if T does not contain F as a facet, then $\Omega(T)$ does not have a pole at F and thus $\text{Res}_F\Omega(T) = 0$. Here, $C_i = T_i \cap F$ for those T_i's with F as facets. By induction hypothesis, we obtain

$$\text{Res}_F(\Omega) = \sum_{F \text{ is a facet of } T_i} \Omega(C_i) = \Omega(F \cap P)$$

as desired. Next, if F is an interior face in the triangulation and $x \in F$, there exists two simplices T_i and T_j containing x. Let $C_i = T_i \cap F$ and $C_j = T_j \cap F$. We see that

$$\text{Res}_F(\Omega)(x) = \text{Res}_F\Omega(T_i)(x) + \text{Res}_F\Omega(T_j)(x) = \Omega(C_i)(x) + \Omega(C_j)(x).$$

Since F lies between T_i and T_j, x is oriented differently in C_i and C_j. This means the above sum vanish and Ω won't have poles at an interior face F.

As a result, $\Omega = \sum \Omega(T_i)$ only has poles at the boundary component of P and that the residues satisfy induction hypothesis. So $\Omega = \Omega(P)$. □

2. Volume of the dual polytope and Filliman duality

2.1. **Compute the volume of the dual.** Let P be a projective polytope. Its dual is defined as

$$P^\vee := \{Y \in \mathbb{P}^m \mid X \cdot Y \geq 0 \text{ for all } X \in P\}.$$

In affine coordinates, $P \in \mathbb{R}^m$, and we usually write

$$P^\vee := \{y \in \mathbb{R}^m \mid x \cdot y \leq 1 \text{ for all } x \in P\}.$$

The main purpose of this section is to establish the following theorem.

Theorem 2.1. *Let P be a polytope. Then for $x \in P_{>0}$,*

$$\Omega(P)(x) = \text{Vol}((P - x)^\vee) dx_1 \cdots dx_n.$$

Let's first calculate the volume. Readers are welcome to visualize via Figure 2 for the following calculation. Assume for simplicity that $0 \in P$. Consider the normal fan of P in the dual space and choose a normal vector w_F for each facet F of P. For a vertex $v \in V(P)$, the rays $\{w_F : v \in F\}$ generate a cone C_v. For a fixed x and a vertex $v \in V(P)$, in the dual space, we can define the following halfspace

$$H^+_{v-x} = \{y \in \mathbb{R}^m \mid \langle y, v - x \rangle \leq 1\}$$

that contains the origin. We then have

$$\text{Vol}((P - x)^\vee) = \sum_{v \in V(P)} \text{Vol}(C_v \cap H^+_{v-x}).$$

Assume for a moment that $S = C_v$ is a simplicial cone with vertex ray $w_1, \ldots, w_m$. Let a_S be the determinant of $w_1, \ldots, w_m$. The ray w_i intersects the hyperplane H_{v-x} at $w_i / \langle w_i, v - x \rangle$. Thus, the volume has the form

$$\text{Vol}(C_v \cap H^+_{v-x}) = a_S \prod_{i=1}^m \frac{1}{\langle w_i, v - x \rangle}.$$

Now if C_v is not simplicial, we need to triangulate C_v into simplicial cones. Let $T(C_v)$ be such a triangulation. View each $S \in T(C_v)$ as a set of m facets of P. We then have

$$\mathrm{Vol}(C_v \cap H_{v-x}^+) = \sum_{S \in T(C_v)} a_S \prod_{F \in S} \frac{1}{\langle w_F, v - x \rangle}.$$

As such, we obtain a formula for the volume of the dual

$$\mathrm{Vol}((P-x)^\vee) = \sum_{v \in V(P)} \mathrm{Vol}(C_v \cap H_{v-x}^+) = \sum_{v \in V(P)} \sum_{S \in T(C_v)} a_S \prod_{F \in S} \frac{1}{\langle w_F, v - x \rangle}.$$

Proof of Theorem 2.1. Let

$$\Omega = \mathrm{Vol}((P-x)^\vee) dx_1 \cdots dx_n = \sum_{v \in V(P)} \sum_{S \in T(C_v)} a_S \prod_{F \in S} \frac{1}{\langle w_F, v - x \rangle} dx_1 \cdots dx_n.$$

We want to show that $\Omega = \Omega(P)$. Clearly Ω doesn't have poles inside P. Let us take a facet G of P and compute $\mathrm{Res}_G \Omega$. We now view x as a point in the interior of G. Then $\langle w_F, v - x \rangle$ vanishes (recalling that $v \in F$) if and only if $F = G$. And thus we need only sum over vertices $v \in V(G)$.

By a linear change of coordinates, we can assume that $G = \{x_n = 0\}$ and $w_G = e_n$, the coordinate vector. For $v \in V(G)$, $\langle w_G, v - x \rangle = x_n$. We have

$$\begin{aligned} \mathrm{Res}_{x_n} \Omega &= \sum_{v \in V(G)} \sum_{S \in T(C_v), G \in S} a_S \cdot \mathrm{Res}_{x_n} \prod_{F \in S, F \neq G} \frac{1}{\langle w_F, v - x \rangle} dx_1 \cdots dx_{n-1} \frac{dx_n}{x_n} \\ &= \sum_{v \in V(G)} \sum_{S \in T(C_v), G \in S} a_S \prod_{F \in S, F \neq G} \frac{1}{\langle w_F, v - x \rangle} dx_1 \cdots dx_{n-1}. \end{aligned}$$

For those $S \in T(C_v)$ with $G \in S$, $\{S \setminus G\}$ forms a triangulation of the cone at v inside $\{x_n = 0\}$. As $w_G = e_n$, we have $a_S = a_{S'}$ by definition, where $S' = S \setminus G$. Comparing the above formula for the volume of the dual, we conclude that $\mathrm{Res}_{x_n} \Omega = \mathrm{Vol}((G-x)^\vee) dx_1 \cdots dx_{n-1}$, which is $\Omega(G)$ by induction hypothesis. By definition for the canonical form, we conclude that $\Omega = \Omega(P)$ as desired. □

An example is done in Section 3.

2.2. **Adjoint of polytopes.** In this section we present work by Warren [3]. Everything here is largely similar to what has happened in Section 2.1. We are just going to present the material in a slightly different language, skipping the proofs, which are the same as above. We primarily think about polytopes in terms of polyhedral cones in $\mathbb{R}^{m+1}$. Let C be a polyhedral cone and we fix a set of vertex rays $V(C)$ that lie in the same hyperplane (possibly $x_0 = 1$). For a simplicial cone S, let a_S be its normalized volume:

$$a_S = \langle v, n_F \rangle a_F$$

where F is a facet, v is opposite of F and n_F is the unit normal.

Definition 2.2. Let $T(C)$ be a triangulation of C by simplicial cones. Then the adjoint of C is

$$A_C(x) = \sum_{S \in T(C)} a_S \prod_{v \in V(C) - V(S)} (v \cdot x).$$

So if S is a simplicial cone, $A_S(x) = a_S$ is a constant.

Lemma 2.3. *If a simplicial cone S has maximal dimension, then for any linear function $L(x)$,*

$$L(x)A_S(x) = \sum_{F \in F(S)} L(n_F)A_F(x)\langle v, x\rangle.$$

Theorem 2.4. *If C has maximal dimension, then for any linear function $L(x)$,*

$$L(x)A_C(x) = \sum_{F \in F(C)} L(n_F)A_F(x) \prod_{v \in V(C)-V(F)} \langle v, x\rangle.$$

Theorem 2.4 shows that the adjoint is defined independent of the triangulation.

Proposition 2.5. *The canonical form and the adjoint are related in the following way:*

$$\Omega(C) = \frac{A_{C^\vee}(x)}{\prod_{F \in C}\langle n_F, x\rangle} dx_1 \cdots dx_n.$$

2.3. **Filliman duality.** Intuitively, the goal of this section is to convey the idea that "dualization of polytopes commutes with triangulation". In the case of triangulation by simplices, it is known as Filliman duality [2].

Theorem 2.6. *Identify P with its characteristic function. Let $P = \sum T_i$ be a (signed) triangulation of P. Then $P^\vee = \sum_i T_i^\vee$ is a (signed) triangulation of $P^\vee$.*

A roundabout way to show that $P^\vee$ and $\sum_i T_i^\vee$ have the same volume can be done via the technology of canonical forms. This is because

$$\begin{aligned}
\Omega_P(x) =& \mathrm{Vol}((P-x)^\vee) dx_1 \cdots dx_n, \\
\Omega_P(x) =& \sum_i \Omega_{T_i}(x) \\
=& \sum_i \mathrm{Vol}((T_i - x)^\vee) dx_1 \cdots dx_n.
\end{aligned}$$

Let's discuss a bit more about Filliman duality, where each T_i is a simplex. The sign of each T_i is determined by the number of facets F of T_i such that O lies on the different side as the simplex T_i. Consider the following example in Figure 1. The sign of $A^\vee$ is positive and the signs of $B^\vee$ and $C^\vee$ are negative.

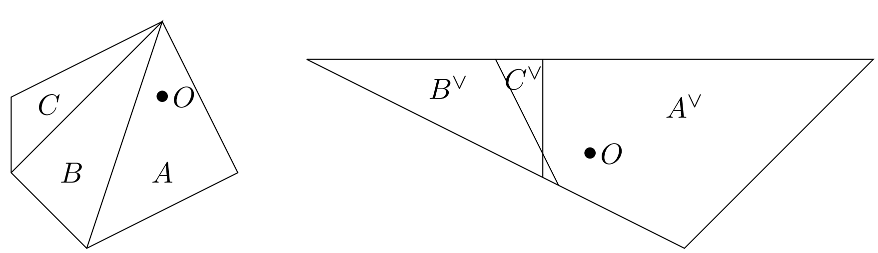

FIGURE 1. A polytope and its Filliman dual

3. An actual example

Consider the following quadrilateral in Figure 2. I would love to do a pentagon but the calculation is too much for the last method. In this case, we have

$$\Omega(P) = \frac{16 - 2x_1 - 6x_2}{(1-x_2)(2-x_1-x_2)(2-x_1+3x_2)(1+x_1)} dx_1 \wedge dx_2.$$

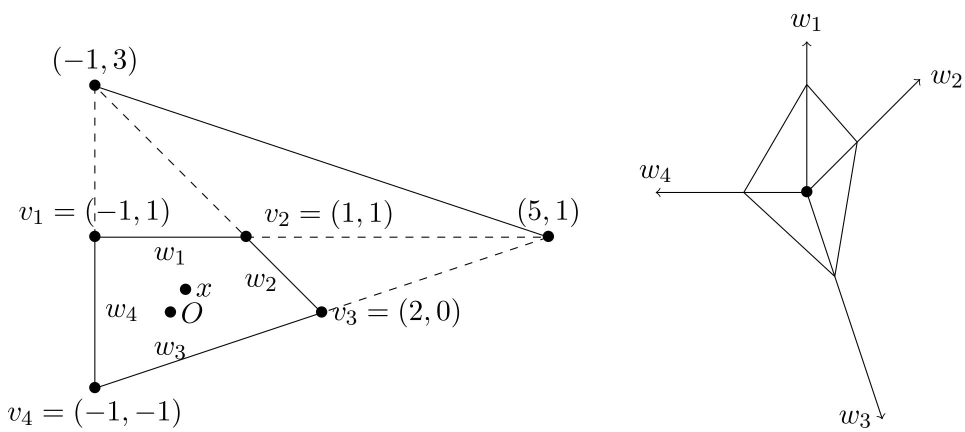

FIGURE 2. A quadrilateral P and its dual $(P - x)^\vee$

3.1. Method 1: observation. Write $\Omega(P) = f dx_1 \wedge dx_2 / \prod \text{edges}$. Since $\Omega(P)$ cannot have a double pole at $(-1, 3)$ and $(5, 1)$, and by degree counting we know $\deg f = 1$ so f must pass through $(-1, 3)$ and $(5, 1)$. We can then take residue and determine the constant.

3.2. Method 2: triangulation P. Let's triangulate P as $T_1 \cup T_2$ via the diagonal $v_2 v_4$, where T_1 contains v_1 and T_2 contains v_3. We have

$$\Omega(T_1) = \frac{c}{(1-x_2)(1+x_1)(x_1-x_2)} dx_1 \wedge dx_2.$$

To figure out the constant c, let's compute its residue (or one could just take the determinant of the lines). We have

$$\begin{aligned} \mathrm{Res}_{x_1=x_2} \Omega(T_1) =& \mathrm{Res}_{x_1=x_2} \left(-\frac{c}{(1-x_2)(1+x_1)} dx_1 \wedge \frac{d(x_1-x_2)}{(x_1-x_2)} \right) \\ =& -\frac{c}{(1-x_2)(1+x_1)} dx_1 \Big|_{x_1=x_2} \\ =& -\frac{c}{(1-x_1)(1+x_1)} dx_1 \end{aligned}$$

so $c = \pm 2$. Take $c = -2$. Similarly,

$$\Omega(T_2) = \frac{c'}{(2-x_1-x_2)(2-x_1+3x_2)(x_1-x_2)} dx_1 \wedge dx_2.$$

We computed

$$\begin{aligned}\mathrm{Res}_{x_1=x_2}\Omega(T_2) =&\mathrm{Res}_{x_1=x_2}\left(\frac{c'}{(2-x_1-x_2)(2-x_1+3x_2)(x_1-x_2)}dx_1\wedge dx_2\right)\\ =&-\frac{c'}{(2-2x_1)(2+2x_1)}dx_1\end{aligned}$$

so $c' = \pm 8$. Take $c' = 8$ so that the orientation is compatible with $c = -2$. Finally, $\Omega(P) = \Omega(T_1) + \Omega(T_2)$ and one can easily check that this is what we need.

3.3. **Method 3: volume of the dual.** We can choose our normal vectors that correspond to facets of P as $w_1 = (0,1)$, $w_2 = (1,1)$, $w_3 = (1,-3)$, $w_4 = (-1,0)$. Then the contribution from each vertex v_i of P is listed as follows

$$\begin{aligned}
v_1 : 1\cdot \frac{1}{\langle w_1, v_1 - x\rangle}\frac{1}{\langle w_4, v_1 - x\rangle}dx_1\wedge dx_2 &= \frac{1}{(1-x_2)(1+x_1)}dx_1\wedge dx_2,\\
v_2 : 1\cdot \frac{1}{\langle w_2, v_2 - x\rangle}\frac{1}{\langle w_1, v_2 - x\rangle}dx_1\wedge dx_2 &= \frac{1}{(2-x_1-x_2)(1-x_2)}dx_1\wedge dx_2,\\
v_3 : 4\cdot \frac{1}{\langle w_3, v_3 - x\rangle}\frac{1}{\langle w_2, v_3 - x\rangle}dx_1\wedge dx_2 &= \frac{4}{(2-x_1+3x_2)(2-x_1-x_2)}dx_1\wedge dx_2,\\
v_4 : 3\cdot \frac{1}{\langle w_4, v_4 - x\rangle}\frac{1}{\langle w_3, v_4 - x\rangle}dx_1\wedge dx_2 &= \frac{3}{(1+x_1)(2-x_1+3x_2)}dx_1\wedge dx_2.
\end{aligned}$$

The sum gives us the canonical form of P.

References

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