Scattering Amplitudes in φ^3 theory

The main reference for these notes is https://arxiv.org/abs/1711.09102.

1 Kinematic Scattering

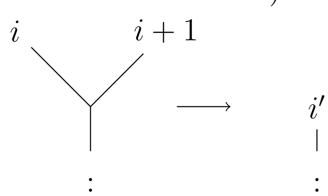
Our first goal is to identify the scattering amplitude in φ^3 theory. The ambient space (the domain of the scattering amplitude) will be the configuration space $\mathcal{K}_n$ of momenta p_i of n massless particles. We will use the Mandelstam variables $s_I = (\sum_{i \in I} p_i)^2$ as (non-independent) coordinates, and we fix the notation $X_{i,j} = s_{[i,j]}$ for $i < j$. This space is spanned by the $s_{i<j}$, with the linear relations $\sum_{j \neq i} s_{i<j} = 0$ for $1 \leq i \leq n$ coming from the on-shell, massless, and momentum conservation conditions, so it has dimension $\frac{n(n-3)}{2}$.

Note that we have the equation $s_{i<j} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}$, and that there are $\frac{n(n-3)}{2}$ nonzero $X_{i,j}$ (all the $X_{i,i+1 \mod n}$ vanish), so the nonzero $X_{i,j}$ form a basis for $\mathcal{K}_n$, so these form independent coordinates for $\mathcal{K}_n$. Now we can define the *planar scattering form.*

Our form will be written as a sum over cubic trees with n leaves corresponding our n (cyclically ordered) particles. For simplicity, we use the cyclic ordering $1 < 2 < \cdots < n < 1$. (Roughly speaking, these graphs correspond to tree level Feynman diagrams.) Note that planarity with a cyclic order is stronger than planarity. In particular, we require that the graph can be drawn as a planar graph when the n vertices are arranged cyclically (say on the boundary of an n-gon).

Lemma 1. *There is a bijection between triangulations of the n-gon and planar cubic trees with n cyclically ordered leaves.*

Proof. We construct a map from planar cubic trees with n cyclically ordered leaves to triangulations. The proof is by induction. If $n = 3$, then we are done because there is only one such tree ($K_{1,3}$), and only one triangulation of the 3-gon. Otherwise, let $n > 3$. Because of our hypotheses, there must be some indicex i such that the edges adjacent to leaves $i, i+1$ meet. In the tree, make the corresponding replacement (keeping the same relative ordering of the now $n-1$ leaves):



and truncate the n-gon by the corresponding triangle $\{(i,i+1),(i+1,i+2),(i,i+2)\}$. What remains is now a planar cubic tree with $n-1$ cyclically ordered leaves, and an $n-1$-gon. By induction, there is a unique triangulation of the $n-1$ gon coming from this tree, and we add back the removed triangle to get a full triangulation of the n-gon. We leave it as an exercise to prove that this is a bijection. □

Corollary 2. *The internal (not adjacent to any leaves) edges of a planar cubic tree (these seem to be called propagators in the physics literature) with n cyclically ordered leaves can be labelled by pairs ij corresponding to diagonals in the n-gon, and there are $n-3$ internal edges on any such tree.*

We now have all the necessary data to define the planar scattering form, $\Omega_n^{(n-3)}$.

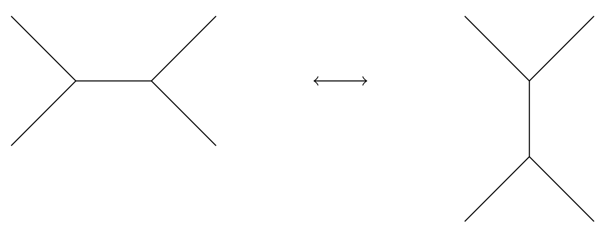
Definition 3. Fix an ordering of the interior diagonals (ij) of an n-gon. Then:

$$\Omega_n^{(n-3)} = \sum_T \mathrm{sign}(T) \bigwedge_{a=1}^{n-3} d\log(X_{i_a,j_a})$$

where the sum is over planar cubic trees with n cyclically ordered leaves, and the wedge is over (some ordering of) the interior edges of the corresponding triangulation of the n-gon. The sign of such a tree is determined by comparing the fixed order of the interior diagonals to the order in which they appear in the wedge (i.e. it is the sign of the permutation which reorders the wedge factors to be consistent with the fixed order).

Remark 4. Depending on the choice of fixed ordering, we may alter $\Omega_n^{(n-3)}$ by a sign. We will be uninterested in this choice. This choice of signs will guarantee that the form is invariant under rescaling the coordinates, and also gives the following relation.

When two planar cubic trees with n cyclically ordered leaves are related by a mutation (described later), their signs differ. A mutation is:



or in terms of the n-gon, exchanging a diagonal in a quadrilateral for the other diagonal. From this description, we should also see the factorization property of the scattering form:

$$\Omega^{(n-3)}(1,\dots,n) \xrightarrow{X_{1,m}\to 0} \Omega^{(m-3)}(1,\dots,m-1,I) \wedge \frac{dX_{1,m}}{X_{1,m}} \wedge \Omega^{(n-m-1)}(I,m,\dots,n)$$

Example 5.

$$\Omega^{(1)}(1,2,3,4) = d\log(X_{1,3}) - d\log(X_{2,4}) = d\log(\frac{X_{1,3}}{X_{2,4}})$$

Alternatively, we could have chosen the other ordering to get:

$$\Omega^{(1)}(1,2,3,4) = -d\log(X_{1,3}) - d\log(X_{1,3}) = d\log(\frac{X_{2,4}}{X_{1,3}})$$

so this object is invariant up to a total sign.

2 Associahedron

The first object we will discuss is a family of polytopes known as *associahedra*. The (type A_n) associahedron is a polytope $\mathcal{A}_n$ ($n \geq 3$) whose face lattice (throughout these notes, face lattices will have bottom element $\widehat{0}$ removed) is the (graded) poset P_n of regular subdivisions of an n-gon with respect to reverse inclusion: for subdivisions S_1, S_2, $S_1 \leq S_2$ iff $S_1 \supset S_2$. We can immediately deduce basic properties of $\mathcal{A}_n$ using this description of its face lattice:

Facts 1. 1. $\dim(\mathcal{A}_n) = n - 3$

2. $\mathcal{A}_n$ is a simple polytope (each edge is adjacent to exactly $n-3$ edges or is contained in exactly $n-3$ facets).

3. A codimension d face of $\mathcal{A}_n$ corresponds to a subdivision of the n-gon with exactly d diagonals/interior edges.

4. For $F_d \subset \mathcal{A}_n$ a face of codimension d, there exist d integers $m_1, \ldots, m_d$, defined uniquely up to permutation, such that the face lattice of F_d is $P_{m_1} \times \cdots \times P_{m_d} \times P_{n+2d-m_1-\cdots-m_d}$.

4'. For $F \subset \mathcal{A}_n$ a facet, there is a unique m such that the face lattice of F is $P_m \times P_{n+2-1}$.

Property 4 defines the poset P_n recursively, i.e. by adjoining a top element $\widehat{1}$ to the to the union of all the order ideals generated by corank 1 elements (not the disjoint union – they should be glued along the order ideal of their intersection). In particular, any polytopal realization of $\mathcal{A}_n$ must satisfy the first three properties as well, since they depend only on the face lattice.

3 Kinematic Associahedron

We will now try to realize the associahedron $\mathcal{A}_n$ in the space $\mathcal{K}_n$. Note that $\dim(\mathcal{A}_n) = n-3$ while $\dim(\mathcal{K}_n) = \frac{n(n-3)}{2}$, so $\mathcal{A}_n$ must necessarily live in a subspace of $\mathcal{K}_n$.

First, we define $\Delta_n \subset \mathcal{K}_n$ by the inequalities $X_{i,j} \geq 0$ for all $1 \leq i < j \leq n$, i.e. Δ_n is the positive orthant in $\mathcal{K}_n$. We further constrain our situation to lie in the intersection of half-spaces H_n defined by $0 < c_{ij} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = -s_{ij}$ for every non adjacent $1 \leq i < j \leq n-1$, where the c_{ij} are positive constants.

Proposition 1. *$Q_n = \Delta_n \cap H_n$ is a polytope realizing the associahedron $\mathcal{A}_n$*

Proof. Note first that Q_n is bounded. (This can be seen by choosing some direction $v \in H_n$ in which Q_n might possibly be unbounded, and then finding a c_{ij} which would be forced to become negative, i.e. find some c_{ij} such that the dot product with v is negative.)

Next, note that H_n contributes no faces (other than the whole polytope). In this sense, we could consider Δ_n as a cone over Q_n. In particular, all the facets of Q_n come from the equalities $X_{i,j} = 0$. As argued before, it suffices to show that each facet has the structure of a product associahedron. For this, we use the following claim:

Faces of a facet $X_{i,j} = 0$ are determined by setting additional $X_{k,l} = 0$ for $(k < l)$ any diagonal that does not cross $(i < j)$. To see this, let (ij) cross (kl), so that $i < j < k < l$. Then summing the defining equations for H_n over the indices $i \leq a < k$ and $j \leq b < l$ gives:

$$\sum_{i \leq a < k, j \leq b < l} c_{ab} = X_{i,j} + X_{k,l} - X_{k,j} - X_{i,l}$$

since this is a telescoping sum. In particular, if both $X_{i,j} = X_{k,l} = 0$ and $(ij), (kl)$ cross, then the above simplifies to

$$\sum_{i \leq a < j, k \leq b < l} c_{ab} = -X_{k,j} - X_{i,l}$$

In particular, the left hand side is positive, while the right hand side is nonpositive, a contradiction. Now, to show the product structure, we proceed by induction. When $n = 1$, Q_1 is a point, so the base case is trivial. For general n the faces of $Q_n \cap \{X_{i,j} = 0\}$ are given by additionally intersecting with $X_{k,l} = 0$ for diagonals (kl) not crossing (ij). In particular, any face of $Q_n \cap \{X_{i,j} = 0\}$ is given by a partial triangulation of the polygons $(1, 2, \ldots, i, j, \ldots n)$ and $(i, i+1, \ldots, j)$, so the face lattice of Q_n is isomorphic to the product lattice $P_{n-(j-i-1)} \times P_{j-i+1}$. Then by property (4'), this determines the entire face lattice of Q_n, which must thus be the poset P_n. (Actually, we need to show property (4), but this should be a straightforward generalization of above. $\square$

Having produced this polytope, we may now ask what its canonical form is. By property (2), Q_n is simple, and in this case we have an easier way to write down the canonical form:

Lemma 2. *For a simple (projective) polytope Q, its canonical form can be written as a sum over the vertices:*

$$\Omega(Q) = \sum_{v \in Q(0)} \text{sign}(v) \bigwedge_{a=1}^{m} d\log(Y \cdot W_a)$$

where Y is an input determining the measure on the dual projective space in which Q^* lives, and the wedge is over the m facets W_a adjacent to v (guaranteed by simplicity). The sign again comes from choosing an ordering of all the facets, and comparing the order of the wedge product to the fixed chosen order.

Proof. We proceed by induction, using the properties of the canonical form (in particular, that it is determined by its poles and residues, and is unique up to scaling).

From the definition, we can see the poles along all the facets from $d\log(Y \cdot W_a)$, since facets are defined by $Y \cdot W_a = 0$. Furthermore, the residue along any such facet is exactly

$$\sum_{v \in \{Y \cdot W_a = 0\}(0)} \text{sign}(v) \bigwedge_{a=1}^{m} d\log(Y \cdot W_a)$$

which by induction is the correct form. Hence we are done. $\square$

In particular, we have the corollary:

Corollary 3. *The canonical form of $Q_n = H_n \cap \Delta_n$ is*

$$\Omega_n = \sum_{v \in Q_n(0)} \text{sign}(v) \bigwedge_{a=1}^{m-1} d\log(X_{i_a, j_a})$$

4 Relating the two forms

Finally, we would like to understand the relationship between the planar scattering form and the canonical form of the associahedron. At this point, the planar scattering form $\Omega_n^{(n-3)}$ is defined on all of $\mathcal{K}_n$, whereas our canonical form is only defined on H_n. However, we might have the following hope: the pullback of $\Omega_n^{(n-3)}$ to H_n is equal to the canonical form of Q_n (say, up to a sign). However, this follows from the fact that the ordering on internal edges of planar cubic trees can be chosen consistently with an order on the diagonals of an n-gon.