

OVERVIEW: POSITIVE GEOMETRIES AND AMPLITUDES

1. PHYSICAL MOTIVATION

In quantum field theory, one computes *scattering amplitudes* [EH], which are probability amplitudes for elementary particle interactions. Among a number of mathematical formalisms in quantum field theory, *Feynman diagrams* are perhaps the best for explicit computations that can be compared to experiment. To set things up, one first has to choose a quantum field theory, which amounts to writing down a Lagrangian, which might look like

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\phi^3 \quad \phi^3\text{-theory}$$

or

$$\mathcal{L} = -\frac{1}{4}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\tilde{D} - m)\psi \quad \text{Yang-Mills theory.}$$

Writing down a Lagrangian involves picking *fields*, which corresponds to choosing a collection of particles. Mathematically, fields are sections of various bundles on space-time ($= \mathbb{R}^4$). In ϕ^3 -theory, ϕ is simply a scalar-valued function. In Yang-Mills theory, $F_{\mu\nu}$ depends on a connection A_μ on a G -bundle, where G is the gauge group. The actual terms in the Lagrangian dictate how the particles interact, and in turn which Feynman diagrams contribute to the answer.

Having chosen the quantum field theory, the scattering amplitude $\mathcal{A}(p_1, \dots, p_n)$ for n -particles is a function of n momentum four-vectors, together with various other data (e.g. what types of particles are involved and other data such as polarization vectors). Feynman diagrams give a formal expansion

$$(1) \quad \mathcal{A}(p_1, \dots, p_n) = \mathcal{A}(p_1, \dots, p_n)_{\text{tree}} + \mathcal{A}(p_1, \dots, p_n)_{1\text{-loop}} + \dots$$

The first term is of the form

$$(2) \quad \mathcal{A}(p_1, \dots, p_n)_{\text{tree}} = \sum_{\text{Feynman diagrams } D \text{ that are trees}} \text{weight}(D)$$

where the weight is written as a product of vertices and edges, and for our purposes we may assume is a rational function in quantities derived from the momentum vectors p_i . The next term $\mathcal{A}(p_1, \dots, p_n)_{1\text{-loop}}$ is a sum over Feynman diagrams with one loop, but now the contribution is, roughly speaking, an integral over a function of the p_i . There are various mathematical issues with the convergence of each integral, and also with the meaning of the infinite sum (1).

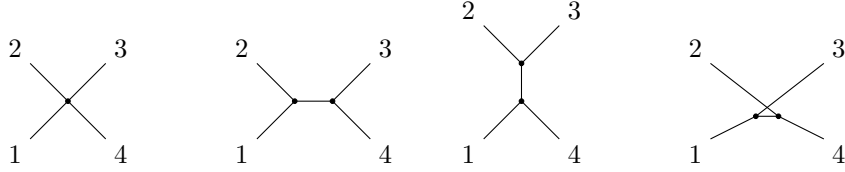
From the practical physical perspective, Feynman diagrams give all the computational tools necessary to compare with particle accelerator experiments. Indeed, various theories such as QED or QCD have been confirmed to high precision since the 1960s. A sort of revolution in the subject happened beginning in the late 1980s when Parke and Taylor [PT] discovered that for gluon (massless-particle) scattering in Yang-Mills theory the tree-level (color-ordered) amplitude was much simpler than the summation would make it appear, and indeed one obtains a formula

$$(3) \quad \mathcal{A}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}.$$

Here the exponents \pm indicate the *helicity* of the gluons, and the quantities $\langle ij \rangle$ is spinor-helicity formalism for certain functions of the p_i . This formula is remarkable because so many Feynman diagrams contribute to the full amplitude (which is a sum of (3) over permutations), as tabulated in [MP]:

n	4	5	6	7	8	9	10
# Feynman diagrams	4	25	220	2485	34300	559405	10525900

For $n = 4$ the four Feynman diagrams are:



(The last picture is a non-planar tree.)

One recent general theme in the theoretical part of scattering amplitudes is to determine whether $\mathcal{A}_{\text{tree}}$ (and other related functions) in various theories can be *uniquely specified* by constraints such as the location of its poles. These constraints are expected to be determined by combinatorial and geometric input. The notion of a **positive geometry** was defined to capture the kind of combinatorics and geometry that (hopefully) appears.

2. DEFINITION OF POSITIVE GEOMETRY

Let $d \geq 0$. A d -dimensional *positive geometry* is a pair $(X, X_{\geq 0})$, where

- X is a d -dimensional normal, irreducible, complex projective variety, defined over \mathbb{R} , and
- $X_{\geq 0} \subset X(\mathbb{R})$ is a semialgebraic subset, with the assumption that $X_{>0} := \text{Int}(X_{\geq 0})$ is a d -dimensional oriented real manifold and $X_{\geq 0} = \overline{X_{>0}}$,

satisfying the following recursive condition. Let $W \subset X$ be the Zariski-closure of $\partial X := X_{\geq 0} \setminus X_{>0}$, and let $C_i \subset W$, $i = 1, 2, \dots, r$ be the irreducible components of W of dimension $d - 1$. Set $C_{i, \geq 0} := C_i \cap X_{\geq 0}$. Then we require that

- (1) Each $(C_i, C_{i, \geq 0})$ is a positive geometry of dimension $d - 1$.
- (2) There exists a unique nonzero rational top-form $\Omega(X, X_{\geq 0})$, called the *canonical form*, on X whose poles are all simple and are exactly along the C_i , and we have

$$\text{Res}_{C_i} \Omega(X, X_{\geq 0}) = \Omega(C_i, C_{i, \geq 0}).$$

For $d = 0$, we declare that (pt, pt) is a positive geometry with Ω equal to either 1 or -1 (taking the role of an orientation).

Example 1. Let $\Delta \subset \mathbb{P}^2$ be a triangle. Then Δ is a positive geometry. In this case, $\partial\Delta$ is the union of three intervals, and W is the union of three distinct (projective) lines. Each $(C_i, C_{i, \geq 0})$ for $i = 1, 2, 3$ is an interval sitting inside a projective line.

Remark 1. There are typically many different positive geometries inside a fixed X . On the other hand, we often omit X from the notation, and use $X_{\geq 0}$ to refer to a positive geometry.

Remark 2. The condition that $\Omega(X, X_{\geq 0})$ is unique implies that X has geometric genus 0. If $d = 1$, then X is a normal projective curve of genus 0, and thus it must be the projective line.

3. EXAMPLES OF POSITIVE GEOMETRIES

3.1. Dimension 1. If $X = \mathbb{P}^1$, then any finite union of closed intervals in $X(\mathbb{R})$ is a positive geometry. For an interval $[a, b]$ (in some affine chart), the canonical form is given by

$$\Omega([a, b]) = -\frac{dx}{x-b} + \frac{dx}{x-a} = \frac{(b-a)}{(x-a)(x-b)} dx.$$

The canonical form of a union of closed intervals is the sum of the canonical forms of those intervals.

Note that $(\mathbb{P}^1, \mathbb{P}^1(\mathbb{R}))$ is not a positive geometry. We have $\partial\mathbb{P}^1(\mathbb{R}) = \emptyset$, but there is no 1-form on \mathbb{P}^1 that has no poles!

3.2. Dimension 2. Suppose $X = \mathbb{P}^2$. Let us consider a closed region R that is bounded by a simple closed curve C , which is union of finitely many pieces each of which is (part of) an algebraic curve. Since curves of degree > 2 in \mathbb{P}^2 have genus > 0 , the region R can only be a positive geometry if all the algebraic curves are linear or quadratic. Almost all such regions R are positive geometries.

For example, let R be the convex hull of the four points $(0, 0), (2, 0), (0, 1), (1, 2)$ in \mathbb{R}^2 . Then R is a positive geometry and we have

$$\Omega(R) = C \frac{y - 4x - 4}{xy(y - x - 1)(2x + y - 4)} dx dy$$

for a constant C . Here note that the denominator expectedly factors into four linear factors corresponding to the facets of R , but the numerator is quite subtle!

As another example, let R be the upper half of the unit disk. Then R is a positive geometry and

$$\Omega(R) = C \frac{1}{y(x^2 + y^2 - 1)} dx dy.$$

Finally, suppose R is the closed unit disk. Then R is *not* a positive geometry, since the boundary is a circle, the whole of $\mathbb{P}^1(\mathbb{R})$, which is not a one-dimensional positive geometry.

3.3. Polytopes. Let $C \subset \mathbb{R}^{n+1}$ be a pointed polyhedral cone of full-dimension. Then the image of $C - \{0\}$ inside $\mathbb{P}^n(\mathbb{R}) \subset \mathbb{P}^n$ is a n -dimensional projective polytope P .

Theorem 1 ([ABL]). *Any projective polytope P is a positive geometry.*

We will investigate the canonical form of polytopes in some detail in this seminar. They serve as the prototypical example of a positive geometry. Amongst many formulae for $\Omega(P)$, we have

$$\Omega(P) = \text{Vol}((P - \mathbf{x})^\vee) dx_1 \wedge \cdots \wedge dx_n$$

where $P \subset \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R})$, and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. This defines a rational differential form on \mathbb{R}^n , which extends to a rational n -form on \mathbb{P}^n .

3.4. Toric varieties. Let X_P be the (normal, projective) toric variety associated to a d -dimensional lattice polytope P . Then it has a natural positive part $X_{P, >0}$, defined to be the positive part $\mathbb{R}_{>0}^d$ of the torus $T(\mathbb{R}) \subset T$ that sits inside X_P . We define the nonnegative part $X_{P, \geq 0} := \overline{X_{P, >0}}$.

Theorem 2 ([ABL]). *$(X_P, X_{P, \geq 0})$ is a positive geometry.*

The canonical form is the natural form $\Omega = d \log x_1 \wedge \cdots \wedge d \log x_n$ on T , extended to a rational top-form on X_P . This is in some sense a simpler example of a positive geometry since Ω has no zeroes on X_P .

3.5. Grassmannians. Let $X = \text{Gr}(k, n)$ be the Grassmannian of k -planes in \mathbb{C}^n . Then Lusztig [Lus] and Postnikov [Pos] have defined the totally nonnegative part $\text{Gr}(k, n)_{\geq 0}$. This is the subspace represented by $k \times n$ matrices all of whose $k \times k$ minors are nonnegative. For $k = 1$, we have $\text{Gr}(1, n)_{\geq 0} = \mathbb{P}_{\geq 0}^{n-1}$ is a $n - 1$ dimensional simplex.

Theorem 3. *$(\text{Gr}(k, n), \text{Gr}(k, n)_{\geq 0})$ is a positive geometry.*

This is quite a non-trivial result, and requires combining results from [Pos, KLS, ABCGPT, Lam]. Part of the non-triviality of this result is that it requires a complete description of all the positive geometries that appear as facets, called *positroid cells* ([Pos]), and properties such as normality of their Zariski-closures, called *positroid varieties* ([KLS]).

More generally, Lusztig has defined the totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety G/P of a split real semisimple group G . It is stated without proof in [ABL] that $(G/P, (G/P)_{\geq 0})$ is a positive geometry.

3.6. Grassmann polytopes and amplituhedra. Let $Z : \mathbb{R}^n \rightarrow \mathbb{R}^{k+m}$ be a linear map, inducing a rational map $Z : \text{Gr}(k, n) \rightarrow \text{Gr}(k, k+m)$. Assuming that Z is well-defined on $\text{Gr}(k, n)_{\geq 0}$, the image $P = Z(\text{Gr}(k, n)_{\geq 0})$ is a Grassmann polytope [Lam], and it is conjectured in [ABL] that P is a positive geometry. When all the $(k+m) \times (k+m)$ minors of Z are positive, we call $A_{n,k,m} = Z(\text{Gr}(k, n)_{\geq 0})$ the *amplituhedron*. The (conjectural) canonical form of $A_{n,k,m}$ is very well-studied due to its relation to super Yang-Mills amplitudes [AT].

More generally, there are higher-loop amplituhedra which are also conjecturally positive geometries.

3.7. Moduli space of points. Let $X = \overline{M}_{0,n}$ be the Deligne-Knudsen-Mumford compactification [DM] of the moduli space of n points on \mathbb{P}^1 . This is a smooth complex projective variety of dimension $n-3$. The open subset $M_{0,n} \subset X$ is the moduli-space of n distinct points on \mathbb{P}^1 . It is isomorphic to a hyperplane arrangement complement. The real part $M_{0,n}(\mathbb{R})$ is a $(n-3)$ -dimensional smooth manifold with $(n-1)!/2$ connected components. We define $(M_{0,n})_{\geq 0} \subset X$ to be the closure in X of one of these connected components. (The group S_n acts on X by permuting the n points, and this action acts transitively on the connected components of $M_{0,n}(\mathbb{R})$.)

Theorem 4. $(\overline{M}_{0,n}, (M_{0,n})_{\geq 0})$ is a positive geometry.

The combinatorics and geometry of the boundary stratification of $\overline{M}_{0,n}$ is very well-studied: each boundary divisor is a product lower-dimensional $\overline{M}_{0,m}$ -s. The canonical form $\Omega((M_{0,n})_{\geq 0})$ is called the Parke-Taylor form in the physics literature. That the residues of $\Omega((M_{0,n})_{\geq 0})$ has the required recursive structure is a consequence, for example, of [AHL, Section 10].

3.8. Ising model, electrical networks. Conjecturally, the space of planar Ising models or the non-negative orthogonal Grassmannian [GP, HW], and the space of electrical networks [Lam18] are positive geometries. These are certain subspaces of the totally nonnegative Grassmannian associated to a (symmetric or skew-symmetric) bilinear form.

4. CONVEXITY, TOPOLOGY, ETC.

4.1. A positive geometry $(X, X_{\geq 0})$ is called *positively convex* if $\Omega(X, X_{\geq 0})$ has constant sign on $X_{>0}$. In other words, the poles and zeros of $\Omega(X, X_{\geq 0})$ do not intersect $X_{>0}$. This is a substitute for the usual notion of convexity for subspaces of \mathbb{R}^n .

4.2. Suppose $(X, X_{\geq 0})$ is a positive geometry and $X_{>0}$ is connected. In many of the examples, $X_{>0}$ is an open ball and $X_{\geq 0}$ is a closed ball (see for example [GKL]). We do not know how the topology of $X_{\geq 0}$ is related to the properties of the canonical form.

4.3. There is a notion of the *pushforward* of the canonical form under a morphism between positive geometries.

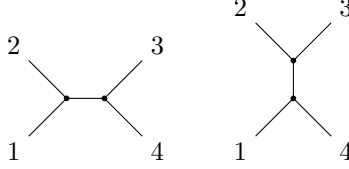
5. EXAMPLES OF INTEGRALS

In most applications of the canonical form, one considers integrals of the form

$$\int_{X_{>0}} (\text{regulator}) \Omega(X_{\geq 0})$$

Since $\Omega(X_{\geq 0})$ has poles along $\partial X_{\geq 0}$, some regulator in the integrand is necessary for the integral to converge. Different regulators appear in different applications.

5.1. **ϕ^3 -amplitudes.** This is the simplest example in this subject that I know of. We consider color-ordered tree-level amplitudes in biadjoint ϕ^3 theory. The Feynman diagrams are planar trees whose leaves are labeled $1, 2, \dots, n$ in cyclic order and interior vertices are degree three. For $n = 4$ the four Feynman diagrams are:



We have n momentum vectors $p_1, \dots, p_n \in \mathbb{R}^4$ satisfying momentum conservation $\sum_{i=1}^n p_i = 0$, and particles are assumed to be massless: $p_i^2 = p_i \cdot p_i = 0$, where the dot product is in Minkowski signature.. The weight associated to a tree is a product of $1/p_e^2$ over all the internal edges e , where $p_e = p_i + p_{i+1} + \dots + p_j$ where i, \dots, j are the leaves on one side of the edge.

In the $n = 4$ case, the two trees contribute $1/s$ and $1/t$ respectively, where $s = (p_1 + p_2)^2 = p_1 \cdot p_2$ and $t = (p_2 + p_3)^2 = p_2 \cdot p_3$. Thus the 4-point amplitude is $\mathcal{A} = 1/s + 1/t$, and this is interpreted as essentially the canonical form of the interval $[0, s + t] \subset \mathbb{R}^1 \subset \mathbb{P}^1$ at the point $x = s$, given by

$$(4) \quad \Omega([0, s + t]) = \left(\frac{1}{x} - \frac{1}{x - (s + t)}\right) dx \stackrel{x=s}{=} \left(\frac{1}{s} + \frac{1}{t}\right) dx.$$

In general, the n -point ϕ^3 -amplitude can be identified with the canonical form of the $(n - 3)$ -dimensional associahedron [ABHY].

5.2. **SYM amplitudes.** This is probably the most striking application of positive geometries, see [ABCGPT, AT, ABL] for more discussion. In this case, we consider the $m = 4$ amplituhedron $A_{n,k,4}(Z)$. The claim, with an abundant amount of evidence, is that tree-level planar $\mathcal{N} = 4$ super-Yang-Mills amplitude is given by an integral of the canonical form $\Omega(A_{n,k,4}(Z))$. This integral turns out essentially to be taking a residue: the form $\Omega(A_{n,k,4}(Z))$ is rational top-form on $\text{Gr}(k, k + 4)$ that also depends rationally on Z . The residue formally produces a rational function that depends on n momentum-twistors Z_1, Z_2, \dots, Z_n (and fermionic variables η_1, \dots, η_n), where $Z_i \in \mathbb{C}^4$. Indeed the rational function descends to a function on $(\mathbb{P}^3)^n/\text{PGL}(4)$.

To see an example of the formalism, let us take $n = 6$ and $k = 1$. Associated to the simplex in \mathbb{P}^4 with vertices a, b, c, d, e , we have the “super function”

$$[a, b, c, d, e] := \frac{\delta^4((Z_a Z_b Z_c Z_d)\eta_e + (Z_b Z_c Z_d Z_e)\eta_a + (Z_c Z_d Z_e Z_a)\eta_b + (Z_d Z_e Z_a Z_b)\eta_c + (Z_e Z_a Z_b Z_c)\eta_d)}{(Z_a Z_b Z_c Z_d)(Z_b Z_c Z_d Z_e)(Z_c Z_d Z_e Z_a)(Z_d Z_e Z_a Z_b)(Z_e Z_a Z_b Z_c)}$$

The indices a, b, c, d, e are (some of the) labels of the n particles involved, and each $(ZZZZ)$ denotes a 4×4 determinant. The Grassmann variables η_i have four components η_{iI} for $I = 1, 2, 3, 4$. The fermionic delta function δ^4 is simply the product of the four components of the expression inside. Expanding this delta-function, the η -s serve as generating function variables. For example, the coefficient of “ $\eta_a^3 \eta_d$ ” is

$$\frac{(Z_b Z_c Z_d Z_e)^3 (Z_e Z_a Z_b Z_c)}{(Z_a Z_b Z_c Z_d)(Z_b Z_c Z_d Z_e)(Z_c Z_d Z_e Z_a)(Z_d Z_e Z_a Z_b)(Z_e Z_a Z_b Z_c)} = \frac{(Z_b Z_c Z_d Z_e)^2}{(Z_a Z_b Z_c Z_d)(Z_c Z_d Z_e Z_a)(Z_d Z_e Z_a Z_b)}$$

which has weight 0 in b, c, e , weight -1 in d , and weight -3 in a .

Now, $\mathcal{A}_{6,1,4}$ is a cyclic polytope P with 6 vertices in \mathbb{P}^4 . It has two triangulations, each using three simplices, related by a *bistellar flip*. The canonical form $\Omega(P)$ is given by $\Omega(P) = \Omega(\Delta_1) + \Omega(\Delta_2) + \Omega(\Delta_3)$ where $\{\Delta_1, \Delta_2, \Delta_3\}$ is either one of these triangulations. The super amplitude in this case is

$$\mathcal{A}_{\text{tree}}(Z, \eta) = [1, 2, 3, 4, 5] + [1, 2, 3, 5, 6] + [1, 3, 4, 5, 6] = [1, 2, 3, 4, 6] + [1, 2, 4, 5, 6] + [2, 3, 4, 5, 6]$$

where the two expressions come from the two triangulations of P .

5.3. **Loop-level amplitudes.** Loop-level amplitudes in SYM theory are in principle integrals of the canonical form of *loop amplituhedra*. However, there are significant challenges in actually performing these integrals. There are numerous connections to the mathematics of Feynman amplitudes, to the theory of motives, and to cluster algebras.

5.4. String amplitudes and Beta functions. This example is based on the positive geometry $(\overline{M}_{0,n}, (M_{0,n})_{\geq 0})$. The n -point (open, gluon, tree-level) string amplitude is given by

$$I_n(s_{ij}) := \int_{(M_{0,n})_{>0}} \Omega((M_{0,n})_{\geq 0}) \prod_{1 \leq a < b \leq n} (ab)^{\alpha' s_{ab}}$$

where if a point in $M_{0,n}$ is represented by a $2 \times n$ matrix (so each column is a point in \mathbb{P}^1), the function (ab) is a 2×2 minor. The expression (ab) is not a well-defined rational function on $M_{0,n}$, but the integrand is well-defined if the momentum conservation equations

$$(5) \quad \sum_{b \neq a} s_{ab} = 0, \quad \text{for all } a$$

holds. Here, α' is a small positive real parameter. The integrand was first written down by Koba and Nielsen [KN].

Many remarkable factorization properties of I_n are encoded in the combinatorics of the positive geometry $(M_{0,n})_{\geq 0}$. For $n = 4$, there are six s_{ab} -s, and four equations (5), so the integral can be written in terms of two independent s_{ab} -s, which we take to be $s = s_{12}$ and $t = s_{13}$. We obtain

$$I_4(s, t) = \int_0^1 \frac{dz}{z(1-z)} z^{\alpha' s} (1-z)^{\alpha' t}$$

which is the Euler Beta function $B(\alpha' s, \alpha' t)$ (called the *Veneziano amplitude* [Ven] in physics), also given by

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

(Note that $\lim_{\alpha' \rightarrow 0} \alpha' I_4 = 1/s + 1/t$, cf. (4).)

5.5. Mirror symmetry and Bessel functions. This example is not directly related to scattering amplitudes, as far as I know. In this case, we consider the positive geometry $(G/B, (G/B)_{\geq 0})$. There is a rational function $f_q : G/B \rightarrow \mathbb{C}$, called the *superpotential*, and it depends on additional quantum parameters $q = (q_i)$. The pair $(G/B, f_q : G/B \rightarrow \mathbb{C})$ is a *Landau-Ginzburg model*, mirror symmetric to the Langlands dual flag variety G^\vee/B^\vee . The integral

$$\Psi((q_i)) = \int_{(G/B)_{\geq 0}} \Omega((G/B)_{\geq 0}) e_q^{f_q}$$

is known as a *Whittaker function*, and mirror symmetry is the statement that $\Psi(q)$ is a solution to the quantum differential equations of G^\vee/B^\vee . In the easiest case, $G = \text{SL}(2)$, and $G/B = \mathbb{P}^1$, and the integral takes the form

$$\Psi(q) = \int_0^\infty \frac{dx}{x} e^{-(x+q/x)}$$

which is essentially a Bessel function. See [Rie, LT, Lam16] for further discussion.

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