LEARNING SEMINAR ON POSITIVE GEOMETRIES SPRING 2020

Recently there has been a flurry of activity in the physics of scattering amplitudes, where *amplitudes* for particles or strings are obtained from spaces called *positive geometries*, with the amplitude thought of informally as a kind of volume. In this approach, amplitudes $A(Z)$ are functions of some momentum variables $Z = (Z_i)$, and typically have the form

$$A(Z) = \int_{X_{\geq 0} \text{ or some other cycle}} \text{integrand}(Z)\Omega(X_{\geq 0}) \tag{1}$$

where $X_{\geq 0}$ is a positive geometry that may depend on Z, and $\Omega(X_{\geq 0})$ is a meromorphic top-form that is called the *canonical form*. In this approach, the Lagrangian formalism and traditional Feynman diagrams in quantum field theory are typically hidden. The goal of this seminar is to understand some of these developments from a mathematical perspective, and to formalize mathematically some of the constructions.

The problem rather naturally splits into two parts:

(1) The construction of positive geometries $X_{\geq 0}$ and the study of their combinatorics and topology and their canonical forms. Examples of positive geometries include usual polytopes, and non-polytopal examples include the positive Grassmannian, the amplituhedron, and the positive part of $M_{0,n}$, the moduli-space of n-points on a Riemann sphere.

(2) The construction of the integrals (1) and the study of the resulting functions. In simple cases the functions $A(Z)$ may be rational functions and the behavior largely combinatorial/algebraic. In other cases, $A(Z)$ has complicated analytic behavior sometimes similar to polylogarithms or hypergeometric functions, have intriguing connections to motives and to cluster algebras.

Goal. Here are some possible goals of the semester:

(a) Formulate precisely some mathematical problems concerning various (conjectural) positive geometries.

(b) Understand, as mathematically rigorously as possible, various formalisms used by amplitude physicists such as notation for differential forms, use of momentum space versus momentum-twistor space, (supersymmetric) delta functions and integrals, and so on.

(c) Learn about symbols, motives, cluster algebras, and so on, that are used to study the functions (1).

The mathematical foundations of some aspects of this subject have not been developed. So part of the task is to develop the necessary mathematical language.

Some possible talks (many topics are multiple talks). I expect to begin with 2-3 talks on understanding canonical forms of polytopes and the toy example of ϕ^3 amplitudes.

1. Definition of positive geometries and canonical forms. `arXiv:1703.04541`
2. Polytopes are positive geometries. `arXiv:1703.04541`
   Warren, Barycentric coordinates for convex polytopes
3. Formalism for differential forms on projective spaces or Grassmannians.
   `arXiv:1703.04541`

(4) Positive Grassmannian is a positive geometry.
`arXiv:math/0609764` `arXiv:1212.5605`
(5) Other totally positive spaces as positive geometries.
Lusztig, Total Positivity in Reductive Groups
(6) Tree amplituhedron and recovering the SYM scattering amplitude.
`arXiv:1312.2007`
(7) Loop amplituhedra
`arXiv:1312.7878` `arXiv:1408.3410` `arXiv:1408.2459` `arXiv:1510.03553`
(8) Momentum space versus momentum-twistor space.
`arXiv:0909.0483` `arXiv:1410.0621`
(9) Supersymmetry: integrals, delta functions, forms on kinematic space.
`arXiv:1212.5605` `arXiv:1704.05069`
(10) Cosmological polytopes. `arXiv:1709.02813`
(11) Basics of Feynman integrals.
(12) Symbols and polylogs, etc..
`arXiv:math/0103059v4` `arXiv:1110.0458`
(13) Cluster structure of loop amplitudes. `arXiv:1305.1617` `arXiv:1710.10953`
(14) $M_{0,n}$ as a positive geometry. `arXiv:math/0606419` `arXiv:1711.09102`
(15) String amplitudes, ϕ^3-amplitudes, and stringy canonical forms.
`arXiv:1711.09102` `arXiv:1912.08707`