Positroid varieties and q,t-Catalan numbers

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Based on joint work with Pavel Galashin

arXiv: 1111.3660 (w. Knutson, Speyer)
arXiv: 1604.06843 (w. Speyer)
arXiv: 1906.03501 (w. Galashin)

$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^2+\dots+q^{n-1}$ $\qquad [n]_q! := [n]_q [n-1]_q \cdots [1]_q$

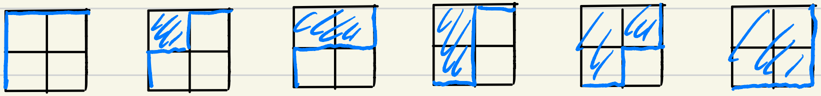
Gaussian polynomial $\qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$

Theorem. $\sum_i \dim H^{2i}(Gr(k,n))\, q^i = \begin{bmatrix} n \\ k \end{bmatrix}_q$ $\qquad$ odd cohomology vanishes

$\#Gr(k,n)(\mathbb{F}_q) \; /\!/$

$= 1 + 1.q + 2q^2 + 1.q^3 + 1q^4$

k=2, n=4



q^{area}

$\binom{4}{2}=6$

$1 + q + q^2 + q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$

- basis of Schubert classes $[\overline{\Omega}_\lambda]$
- hard Lefshetz ⇒ symmetry, unimodality for $Gr(k,n)$

Sylvester 1877 "waiting for a proof for the last quarter century or upwards"

White, Macdonald, Proctor, Stanley, O'Hara, ...

Binomial → Catalan

$\gcd(a,b)=1$

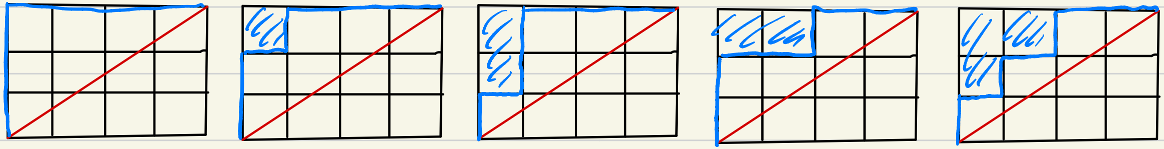
Catalan number $C_n = C_{n,n+1}$ Rational Catalan number $C_{a,b} := \frac{1}{a+b}\binom{a+b}{a}$

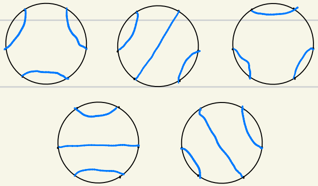
Catalan (1814-1894) Fuss (1755-1826) Grossman 1950

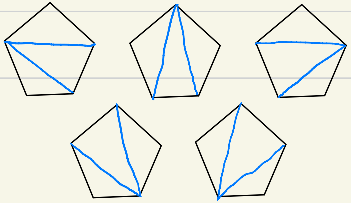
$(a,b)=(n,kn+1)$ Bizley 1954

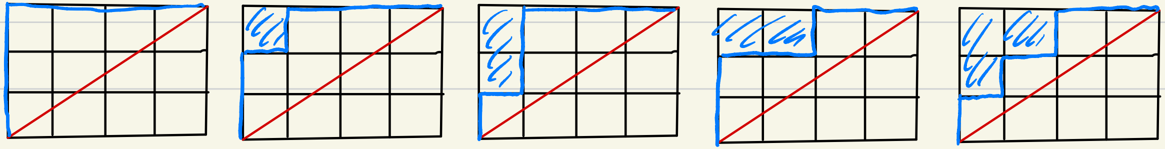
Theorem $C_{a,b} = \#$ rational Dyck paths in $a \times b$ rectangle

$C_3 = C_{3,4} = 5$









$$\sum q^{\text{area}} = 1 + q + q^2 + q^2 + q^3$$

$$\frac{1}{[7]_q}\begin{bmatrix}7\\3\end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6$$

q,t-rational Catalan

$$C_{a,b}(q,t) := \sum q^{\text{coarea}} t^{\text{dinv}}$$

Garsia-Haiman

Haglund

Loehr-Warrington

Gorsky-Mazin

Armstrong-Hanusa-Jones

$$C_3(q,t) = C_{3,4}(q,t) = q^3 + qt + q^2 t + q t^2 + t^3$$

$$C_{a,b}(q,1) = \sum q^{\text{coarea}}$$

$$q^{(a+b-1)/2} C_{a,b}\left(q,\frac{1}{q}\right) = \frac{1}{[a+b]_q}\begin{bmatrix}a+b\\a\end{bmatrix}_q$$

$Gr(k,n) \xleftrightarrow[\text{mirror}]{} (\overset{\circ}{\Pi}_{k,n}, f: \overset{\circ}{\Pi}_{k,n} \to \mathbb{C})$ Landau-Ginzburg mirror.

Positroid top cell $\overset{\circ}{\Pi}_{k,n} = \{\Delta_{12\ldots k} \neq 0 \quad \Delta_{23\ldots k+1} \neq 0 \quad \ldots \quad \Delta_{n12\ldots(k-1)} \neq 0\} \subset Gr(k,n)$

(not a cell)

complement of an anticanonical divisor

$K_{Gr(k,n)} = -nH$

$\overset{\circ}{\Pi}_{k,n}$ is a smooth affine variety of dimension $d = k(n-k)$

Theorem The cohomology of $\overset{\circ}{\Pi}_{k,n}$ is of Hodge-Tate type and mixed Hodge poly is

$\gcd(k,n) = 1$

$$P(\overset{\circ}{\Pi}_{k,n}; q, t) := \sum \dim H^{k,(p,p)}(\overset{\circ}{\Pi}_{k,n}, \mathbb{C})\, q^{\frac{d-k}{2}} t^{p-\frac{k}{2}} = (q^{1/2} + t^{1/2})^{n-1} C_{k,n-k}(q,t)$$

Hodge-Tate: $H^{k,(p,q)} = 0$ for $p \neq q$ where $H^k = \bigoplus_{p,q} H^{k(p,q)}$ Deligne splitting

(mixed Tate)

$\# Gr(k,n)(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ and $\# \Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^n \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$

$$\Downarrow$$

Weird corollary Sample $V \in Gr(k,n)(\mathbb{F}_q)$ uniformly

$\mathrm{Prob}(V \in \overset{\circ}{\Pi}_{k,n}) = \frac{(q-1)^n}{q^n - 1}$ No dependence on k!?!

Th'm mixed hodge poly$(\mathring{X}_{k,n}) = C_{k,n-k}(q,t)$

$\mathring{\Pi}_{k,n} \circlearrowleft T \cong (\mathbb{C}^*)^{n-1}$ free $\qquad \mathring{X}_{k,n} := \mathring{\Pi}_{k,n}/T \overset{\text{birat.}}{\approx}$ configuation space of n points in $\mathbb{P}^{k-1}$

$\text{Poincare poly}(\mathring{X}_{k,n}) = \sum q^{2\,\text{area}}$  odd cohomology vanishes!

$\# \mathring{X}_{k,n}(\mathbb{F}_q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$ What about "Schubert" basis? no cell decomposition!

$a=3 \quad b=4$

Example $\dim = 3\times 4 - 6 = 6$

$\mathring{X}_{3,7}$ Poincare: $1+q^2+2q^4+q^6$ $\quad \#(\mathbb{F}_q): 1+q^2+q^3+q^4+q^6$

Discrepancy: H^4 is not pure.

$\dim H^{4,(4,4)} = 1 \qquad \dim H^{4,(3,3)} = 1$

$Gr(k,n) = \bigsqcup_{\lambda \subset \square} \Omega_\lambda$

Generalization to positroid varieties (Knutson-L.-Speyer)

$\Omega_\lambda \cong \mathbb{C}^{|\lambda|}$ = Schubert cell indexed by λ

$$\mathring{\Pi}_f := \Omega_{\lambda_1} \cap \chi(\Omega_{\lambda_2}) \cap \dots \cap \chi^{n-1}(\Omega_{\lambda_n})$$

roughly $\rightarrow$

$f \in S_n$ $\quad \chi : Gr(k,n) \longrightarrow Gr(k,n)$ cyclic rotation automorphism

Theorem

$P(\mathring{\Pi}_f; q, t)$ = Khovanov-Rosanzky style link invariant.

$\# \mathring{\Pi}_f(\mathbb{F}_q)$ = coefficient of HOMFLY polynomial

Further generalizes to open Richardson varieties in G/B

$\mathring{\Pi}_f$ important in

- mirror symmetry
- Poisson geometry
- cluster algebras
- Frobenius splitting
- total positivity
- Schubert calculus
- . . . . .

$C_{a,b}(q,t) = C_{a,b}(t,q)$

Corollary $\underline{C_{a,b}(q,t)}$ is q,t-symmetric and unimodal

$$C_{3,5}(q,t) = q^4 + q^3 t + q^2 t^2 + q t^3 + t^4 + q^2 t + q t^2$$

Hainan
Carlsson-Mellit : $C_n(q,t)$ symmetry + unimodality $\sim$ Macdonald polynomials

Mellit, Gorsky-Negut : $C_{a,b}(q,t)$ symmetry

Unimodality appears new

Theorem Curious Lefschetz holds for $\mathring{X}_{k,n}$ and $\mathring{\Pi}_f$:

$\exists\ \gamma \in H^{2,(2,2)}(X)$

$$\gamma^{d-p} : H^{p+s,(p,p)}(X) \xrightarrow{\sim} H^{2d-p+s,(2d-p,2d-p)}(X)$$

symmetry around (d,d)

$\Longrightarrow$ unimodality and symmetry of $P(X;q,t)$.

Hausel Rodriguez-Villegas : defined curious Lefschetz.

L. - Speyer : holds for certain cluster varieties

Galashin - L. : $\mathring{\Pi}_f$ are cluster varieties

Koszul duality for $D^b_{(B),m}(\mathcal{F}l_n;\overline{\mathbb{Q}}_\ell)$

constructible along Schubert stratification

mixed

flag variety

$$\mathcal{F}l_n = \coprod_{w\in S_n} X_w$$

Theorem $\Phi : D^b_{(B),m}(\mathcal{F}l_n;\overline{\mathbb{Q}}_\ell) \to D^b_{(B),m}(\mathcal{F}l_n;\overline{\mathbb{Q}}_\ell)$ equivalence

$$\Phi(\Delta_w) = \Delta_{w^{-1}} \qquad \Delta_w := i_!\,\overline{\mathbb{Q}}_{\ell\, X_w}[\ell(w)](\ell(w)/2)$$

where $i: X_w \hookrightarrow \mathcal{F}l_n$

cohom. weight

Induces $\quad \mathrm{Ext}^{*,\bullet}(\Delta_v,\Delta_w) \cong \mathrm{Ext}^{*-\bullet,-\bullet}(\Delta_{v^{-1}},\Delta_{w^{-1}}) \cong \mathrm{Ext}^{*-\bullet,-\bullet}(\Delta_v,\Delta_w)$

can be related to $H^*(\tilde{\Pi}_f)$

Beilinson-Ginzburg-Soergel: original Koszul duality

Bezrukavnikov-Yun: self-duality

What about Lefschetz operator?

Idea of proof:

$\beta_{v,w} =$ 

$H_T^*(\mathring{\Pi}_f)$

$\cong$ ← $\mathring{\Pi}_f \simeq \mathring{R}_{v,w}$ Knutson-L.-Speyer

gives a braid $\beta_{v,w}$

"Schubert ∩ opp. Schub.

H_T^*(open Richardson)

$\cong$ ← (equivariant) localization th'm Beilinson-Bernstein Brylinski-Kashiwara Riche-Soergel-Williamson

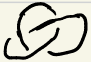
Ext-group in $D_B^b(\mathcal{F}\ell_n)$

$\cong$ ← take $H_B^\bullet$ Soergel, Achar-Riche, Bezrukavnikov-Yun, ...

Calculation with Soergel bimodules

$\cong$ ← Khovanov via Rouquier complexes

KR-homology of a link

$\bullet \leftrightarrow$  trivial $\mathring{\Pi}_{2,4} \longleftrightarrow$  Hopf $\mathring{\Pi}_{3,5} \longleftrightarrow$  trefoil

$\updownarrow$ ← Mellit

$C_{a,b}(q,t)$

recursion involves things that aren't links

Shende-Treumann-Zaslow

Conjectural P=W theorem for $\mathring{X}_{k,n}$ motivated by Shende-Treumann-Williams-Zaslo

There exists a classifies rank one torsion-free sheaves on

// complete curve.

$$\mathring{X}_{a,a+b} \xrightarrow[\text{retract}]{\text{deformation}} J_{a,b}$$

Compactified Jacobian of plane curve singularity $x^a = y^b$

compact, dim $= \frac{(a-1)(b-1)}{2} = \frac{1}{2} \dim \mathring{X}_{k,n}$

singular

$$W_{2k}(H^*(\mathring{X}_{a,a+b})) \xrightarrow{\sim} P_k(H^*(J_{a,b}))$$

Weight

Perverse

Maulik-Yun

Migliorini-Shende

Beauville: $\chi(J_{a,b}) = C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$

Piontkowski: cell decomposition $\longleftrightarrow$ $C_{a,b}(q,1)$ also Lusztig-Smelt
$\Rightarrow$ basis for $H^*(J_{a,b})$ and point count

Oblomkov-Yun: generators for $H^*(J_{a,b})$

Gorsky-Mazin: (conjectural) relation to $C_{a,b}(q,t)$

Gorsky-Oblomkov-Rasmussen-Shende: (conjectural) relation to knot homology
Gorsky-Negut, ...