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Pieri and Cauchy Formulae for Ribbon Tableaux

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Abstract. In [**LLT**] Lascoux, Leclerc and Thibon introduced symmetric functions $\mathcal{G}_\lambda$ which are spin and weight generating functions for ribbon tableaux. This article is aimed at studying these functions in analogy with Schur functions. In particular we will describe:

- a Pieri and dual-Pieri formula for ribbon functions,
- a ribbon Murnaghan-Nakayama formula,
- ribbon Cauchy and dual Cauchy identities,
- and a $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ which sends each $\mathcal{G}_\lambda$ to $\mathcal{G}_{\lambda'}$.

We will show that the ribbon Pieri and Murnaghan-Nakayama rules are formally equivalent in a purely combinatorial manner. We will also connect the ribbon Cauchy and Pieri formulae to the combinatorics of ribbon insertion as studied by Shimozono and White [**SW2**]. In particular we give complete combinatorial proofs for the domino $n = 2$ case.

Résumé. Dans [**LLT**], Lascoux, Leclerc et Thibon ont introduit des fonctions symétriques $\mathcal{G}_\lambda$ qui sont les series formelles pour tableaux des rubans, selon la rotation et le poids. Cet article est visé à l'étude de ces fonctions dans l'analogie avec les fonctions de Schur. En particulier nous décrirons:

- des formules ruban-Pieri et dual-ruban-Pieri,
- une formule de ruban Murnagham-Nakayama,
- les identités ruban-Cauchy et dual-ruban-Cauchy pour fonctions de ruban,
- et un isomorphisme $\mathbb{C}$-algèbre $\omega_n : \Lambda(q) \to \Lambda(q)$ qui envoie chaque $\mathcal{G}_\lambda$ à $\mathcal{G}_{\lambda'}$.

Nous montrerons que les règles Pieri de et Murnagham-Nakayama sont formellement équivalents dans une manière purement combinatoire. Nous connecterons aussi les formules ruban-Cauchy et ruban-Pieri au combinatoire d'insertion des rubans, comme étudié par Shimozono et White [**SW2**]. En particulier, nous donnons les preuves combinatoires complétes pour le cas domino $n = 2$.

Introduction

This abstract is a much shortened version of the paper [**Lam1**]. It has been rewritten with the focus placed on combinatorial aspects. Many results and essentially all the proofs together with the representation theoretic details have been removed.

Let $n \geq 1$ be a fixed integer and λ a partition with empty n-core. In analogy with the combinatorial definition of the Schur functions, Lascoux, Leclerc and Thibon [**LLT**] have defined a family of symmetric functions $\mathcal{G}_\lambda(X;q) \in \Lambda(q)$ by:

$$\mathcal{G}_\lambda(X;q) = \sum_T q^{s(T)} \mathbf{x}^{w(T)}$$

where the sum is over all *semistandard ribbon tableaux* of shape λ, and $s(T)$ and $w(T)$ are the spin and weight of T respectively. The definition of a semistandard ribbon tableau is analagous to the definition of semistandard Young tableaux, with boxes replaced by ribbons (or border strips) of length n. We shall loosely call the functions $\mathcal{G}_\lambda(X;q)$ *ribbon functions*.

When $q = 1$ the ribbon functions become products of usual Schur functions. However, when the parameter q is introduced, it is no longer obvious that the functions $\mathcal{G}_\lambda(X;q)$ are symmetric.

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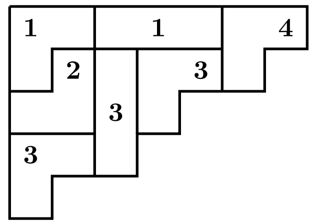

FIGURE 1. A semistandard 3-ribbon tableau with shape $(7,6,4,3,1)$, weight $(2,1,3,1)$ and spin 7.

The main aim of this paper will be to develop the theory of ribbon functions in the same way Schur functions are studied in the ring of symmetric functions. We shall see that the appropriate 'ribbon' analogues of the power sum, homogeneous and elementary symmetric functions is given by the the plethysm

$$f \mapsto f[(1+q^2+\cdots+q^{2n-2})X].$$

We show that this leads to a ribbon Pieri rule in a natural way and also define 'border ribbon strips' which lead to a ribbon Murnaghan-Nakayama rule. These two rules are connected by showing that they are formally equivalent in a combinatorial fashion. The plethysm of the Cauchy kernel leads to a Cauchy and dual-Cauchy identity. We also describe a $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ which sends each skew ribbon function to the ribbon function corresponding to the conjugate.

It is well known that the corresponding formulae are important for Schur functions in representation theory and algebraic geometry.

Much of the interest in the ribbon functions has been focused on the q-Littlewood Richardson coefficients $c^{\mu}_{\lambda}(q)$ of the expansion of $\mathcal{G}_\lambda(X;q)$ in the Schur basis:

$$\mathcal{G}_\lambda(X;q) = \sum_{\mu} c^{\mu}_{\lambda}(q) s_\mu(X).$$

These are q-analogues of Littlewood Richardson coefficients. Using results of Varagnolo and Vasserot [**VV**], Leclerc and Thibon [**LT**] have shown that these coefficients are parabolic Kazhdan-Lusztig polynomials of type A. Results of Kashiwara and Tanisaki [**KT**] then imply that they are polynomials in q with non-negative coefficients. Much interest has also developed in connecting ribbon tableaux and the q-Littlewood Richardson coefficients to rigged configurations and the generalised Kostka polynomials defined by Kirillov and Shimozono [**KS**], Shimozono and Weyman [**SW3**], Schilling and Warnaar [**SchW**] and Shimozono [**Shi**].

To prove that the functions $\mathcal{G}_\lambda(X;q)$ were symmetric Lascoux, Leclerc and Thibon connected them to Fock space representation $\mathbf{F}$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. The crucial property of $\mathbf{F}$ is an action of a Heisenberg algebra H, commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$, discovered by Kashiwara, Miwa and Stern [**KMS**]. In particular, they showed that as a $U_q(\widehat{\mathfrak{sl}}_n) \times H$-module, $\mathbf{F}$ decomposes as

$$\mathbf{F} \cong V_{\Lambda_0} \otimes \mathbb{C}(q)[H_-]$$

where V_{Λ_0} is the highest weight representation of $U_q(\widehat{\mathfrak{sl}}_n)$ with highest weight Λ_0 and $\mathbb{C}(q)[H_-]$ is the usual Fock space representation of the Heisenberg algebra.

In [**Lam1**], the connection between ribbon functions and the action of the Heisenberg algebra is made explicit by showing that the map $\mathbf{\Phi} : \mathbf{F} \to \mathbb{C}(q)[H_-]$ defined by

$$|\lambda\rangle \mapsto \mathcal{G}_\lambda$$

is a map of H-modules, after identifying $\mathbb{C}(q)[H_-]$ with the ring of symmetric functions $\Lambda(q)$ in the usual way. The map $\mathbf{\Phi}$ has the further remarkable property that it changes certain linear maps into algebra maps (for example leading to ω_n). Via the map $\mathbf{\Phi}$, the action of the Heisenberg algebra leads to the ribbon Murnaghan-Nakayama and Pieri rules. Unfortunately, we will not be able to explore this aspect of the subject in this abstract.

We shall also connect our study of ribbon functions to more combinatorial aspects of ribbon tableaux. Using the domino insertion of Barbasch and Vogan, Garfinkle and Shimozon and White [**BV, Gar, SW**] we will give combinatorial proofs of the Pieri and Murnaghan-Nakayama formulae. The Cauchy and dual-Cauchy identities were observed earlier in [**Lam**]. Shimozono and White [**SW2**] have defined a ribbon-Schensted algorithm for $n > 2$ which is also compatible with spin on ribbon tableaux. As we shall discuss, this algorithm gives a combinatorial proof of the first ribbon Pieri formula for $k = 1$, but appears to be insufficient to prove either the Cauchy identity or the higher Pieri rules.

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1. Partitions and Tableaux

A distinguished integer $n \geq 1$ will be fixed throughout the whole article. When $n = 1$, the reader may check that we recover the classical theory of Schur functions. We will use the usual notation and definitions for partitions, compositions, horizontal strips, border strips, standard and semistandard Young tableaux which can be found in [**EC2, Mac**].

Let b be a border strip. The height $h(b)$ is the number of rows in b, minus 1. When a border strip has n squares for the distinguished (fixed) integer n, we will call it a ribbon. The height of the ribbon r will then be called its spin $s(r)$. The reader should be cautioned that in the literature the spin is usually defined as half of this.

Let λ be a partition. Its n-core, obtained from λ by removal of n-ribbons (until we are no longer able to), is denoted $\tilde{\lambda}$. The n-quotient of λ will be denoted $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$. We shall write $\mathcal{P}$ for the set of partitions. We will use $\mathcal{P}_\delta$ to denote the set of partitions λ such that $\tilde{\lambda} = \delta$ for an n-core $\delta = \tilde{\delta}$.

A ribbon tableau T of shape λ/μ is a tiling of λ/μ by n-ribbons and a filling of each ribbon with a positive integer (see Figure 1). We will use the convention that a ribbon tableau of shape λ where $\tilde{\lambda} \neq \emptyset$ is simply a ribbon tableau of shape $\lambda/\tilde{\lambda}$. A ribbon tableau is semistandard if for each i

(1) removing all ribbons labelled j for $j > i$ gives a valid skew shape $\lambda_{\leq i}/\mu$ and,
(2) the subtableau containing only the ribbons labelled i form a *horizontal n-ribbon strip.*

A horizontal n-ribbon strip is a skew shape tiled by ribbons such that the topright-most square of every ribbon touches the northern edge of the shape (see Figure 2). If such a tiling exists, it is necessarily unique. If the numbers occuring in a ribbon tableau are exactly $\{1, 2, \ldots, m\}$, for some m, then the tableau is called standard.

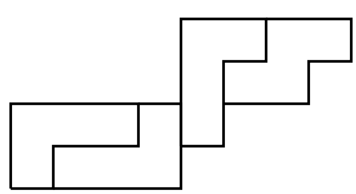

FIGURE 2. A horizontal 4-ribbon strip with spin 5.

We will often think of a ribbon tableau as a chain of partitions

$$\tilde{\lambda} = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

where each μ^{i+1}/μ^i is a horizontal ribbon strip. The partitions μ^i here are not to be confused with the n-quotient of μ.

The spin $s(T)$ of a ribbon tableau T is the sum of the spins of its ribbons. The weight $w(T)$ of a tableau is the composition counting the occurences of each value in T.

All these concepts and statistics on ribbon tableau can be described in terms of the n-quotient (see [**SSW**]).

2. Symmetric Functions

In this section we briefly review some standard notation in symmetric function theory. The reader is referred to [**Mac**] for further details.

Let $\Lambda_{\mathbb{Z}}$ denote the ring of symmetric functions with coefficients in $\mathbb{Z}$. Recall that $\Lambda_{\mathbb{Z}}$ has a distinguished integral basis s_λ known as the Schur functions. Nearly all the results of this paper can be stated in $\Lambda_{\mathbb{Z}}[q]$, but some intermediate steps may require working in $\Lambda = \Lambda_{\mathbb{C}}$ so we will use that as our symmetric function ring from now on. We will write $\Lambda(q)$ for $\Lambda_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}(q)$.

It is well known that the Schur functions s_λ are orthogonal with respect to a natural inner product $\langle\, ,\, \rangle$ on Λ and are unique up to signed permutation. We will denote the homogeneous, elementary, monomial and power sum symmetric functions by h_λ, e_λ, m_λ and p_λ respectively. Recall that we have $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ where $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. Each of $\{p_i\}$, $\{e_i\}$ and $\{h_i\}$ generate Λ. We will write X to mean $(x_1, x_2, \ldots)$. Thus $s_\lambda(X) = s_\lambda(x_1, x_2, \ldots)$.

Let $f \in \Lambda$. We will recall the definition of the plethysm $g \mapsto g[f]$. Write $g = \sum_\lambda c_\lambda p_\lambda$. Then we have

$$g[f] = \sum_\lambda c_\lambda \prod_{i=1}^{l(\lambda)} f(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots).$$

Thus the plethysm by f is the (unique) algebra isomorphism of Λ which sends $p_k \mapsto f(x_1^k, x_2^k, \ldots)$. When $f(x_1, x_2, \ldots; q) \in \Lambda(q)$ for the distinguished element q, we define the plethysm as $p_k \mapsto f(x_1^k, x_2^k, \ldots; q^k)$. Thus plethysm does not commute with spcialising q to a complex number.

For example, the plethysm by $(1+q)p_1$ is given by sending

$$p_k \mapsto (1 + q^k) p_k$$

and extending to an algebra isomorphism $\Lambda(q) \to \Lambda(q)$. In such situations we will write $f[(1+q)X]$ for $f[(1+q)p_1]$.

We will be particularly concerned with the plethysm given by $(1 + q^2 + \cdots + q^{2n-2})p_1$. We will use $\Upsilon_{q,n}$ to denote the map $\Lambda(q) \to \Lambda(q)$ given by $f \mapsto f[(1 + q^2 + \cdots + q^{2n-2})X]$.

3. Ribbon Functions

We will now define the central objects of this paper as introduced by Lascoux, Leclerc and Thibon in [**LLT**].

Definition 3.1. Let λ/μ be a skew partition, tileable by n-ribbons. Define the symmetric functions $\mathcal{G}_{\lambda/\mu} \in \Lambda(q)$ as:

$$\mathcal{G}_{\lambda/\mu}(X; q) = \sum_T q^{s(T)} \mathbf{x}^{w(T)}$$

where the sum is over all semistandard ribbon tableaux T of shape λ/μ and $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. When λ is a partition with non-empty n-core, we write $\mathcal{G}_\lambda$ for $\mathcal{G}_{\lambda/\tilde{\lambda}}$. These functions will be loosely called *ribbon functions*.

The fact that the functions $\mathcal{G}_{\lambda/\mu}$ are symmetric is not obvious from the combinatorial definition. The proof requires the use of the action of the Heisenberg algebra on the Fock space of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$.

THEOREM 3.2 ([**LLT**]). *The functions $\mathcal{G}_{\lambda/\mu}(X;q)$ are symmetric functions.*

DEFINITION 3.3. Let λ/μ be a skew shape tileable by n-ribbons. Then define

$$\mathcal{K}_{\lambda/\mu,\alpha}(q) = \sum_T q^{s(T)},$$

the spin generating function of all semistandard ribbon tableaux T of shape λ/μ and weight α. Similarly let

$$\mathcal{L}_{\lambda/\mu,\alpha}(q) = \sum_T q^{s(T)}$$

summed over all column semistandard ribbon tableaux of shape λ/μ and weight α. A ribbon tableau is column semistandard if its conjugate is semistandard.

Thus $\mathcal{G}_{\lambda/\mu}(X;q) = \sum_\alpha \mathcal{K}_{\lambda/\mu,\alpha}(q)\mathbf{x}^\alpha$. We will now define *border ribbon strips*.

DEFINITION 3.4. A *border ribbon strip* T is a connected skew shape λ/μ with a distinguished tiling by disjoint non-empty horizontal ribbon strips $T_1, \ldots, T_a$ such that the diagram $T_{+i} = \cup_{j\leq i} T_j$ is a valid skew shape for every i and for each connected component C of T_i we have

(1) The shape of $C \cup T_{i-1}$ is not a horizontal ribbon strip. Thus C has to 'touch' T_{i-1} 'from below'.
(2) No sub horizontal ribbon strip C' of C which can be added to T_{i-1} satisfies the above property. Since C is connected, this is equivalent to saying that only the rightmost ribbon of C touches T_{i-1}.

We further require that T_1 is connected. The height $h(T_i)$ of the horizontal ribbon strip T_i is the number of its components. The height $h(T)$ of the border ribbon strip is defined as $h(T) = (\sum_i h(T_i)) - 1$. The size of the border ribbon strip T is then the total number of ribbons in $\cup_i T_i$. A border ribbon strip tableau is a chain $T = \lambda_0 \subset \lambda_1 \cdots \subset \lambda_r$ of shapes such that λ_i/λ_{i-1} has been given the structure of a border ribbon strip. The type of $T = \{\lambda_i\}$ is then the composition α with α_i equal to the size of λ_i/λ_{i-1}.

Define $\mathcal{X}_\nu^{\mu/\lambda}$ as

$$\mathcal{X}_\nu^{\mu/\lambda}(q) = \sum_T (-1)^{h(T)} q^{s(T)}$$

summed over all border ribbon strip tableaux of shape μ/λ and type ν.

Note that this definition reduces to the usual definition of a border strip and border strip tableau when $n = 1$, in which case all the horizontal strips T_i are actually connected.

EXAMPLE 3.5. Let $n = 2$ and $\lambda = (4,2,2,1)$. Suppose S is a border ribbon strip such that S_1 has shape $(7,5,2,1)/(4,2,2,1)$, and thus it has size 3 and spin 1. We will now determine all the possible horizontal ribbon strips which may form S_2. It suffices to find the possible connected components that may be added. The domino $(9,5,2,1)/(7,5,2,1)$ may not be added since its union with S_1 is a horizontal ribbon strip, violating the conditions of the definition. The domino strip $(8,8,2,1)/(7,5,2,1)$ is not allowed since the domino $(8,8,2,1)/(7,7,2,1)$ can be removed and we still obtain a strip which touches S_1.

The legitimate connected horizontal ribbon strips C which can be added are $(7,7,2,1)/(7,5,2,1)$, $(7,5,3,3,2,1)/(7,5,2,1)$ and $(7,5,4,1)/(7,5,2,1)$ as shown in Figure 3. Thus assuming S_2 is non-empty, there are 5 choices for S_2, corresponding to taking some compatible combination of the three connected horizontal ribbon strips above.

EXAMPLE 3.6. As before let $n = 2$. We will calculate $\mathcal{X}_5^{\lambda/\mu}(q)$ for $\lambda = (5,5,2)$ and $\mu = (2)$. The relevant border ribbon strips S are (successive differences of the following chains denote the S_i)

- $(2) \subset (5,5,2)$ with height 0 and spin 5,
- $(2) \subset (5,3,2) \subset (5,5,2)$ with height 1 and spin 3,

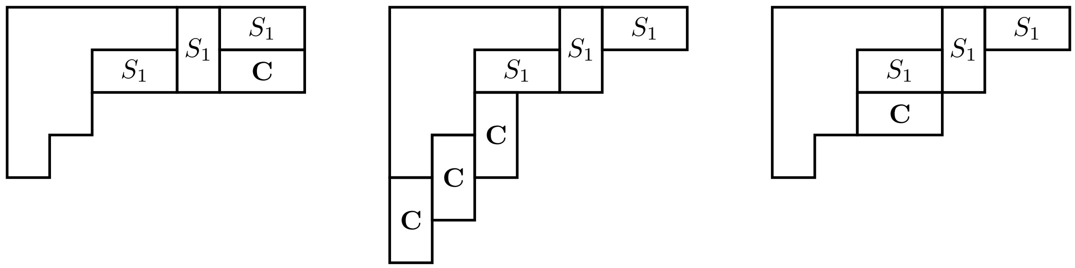

FIGURE 3. Connected horizontal strips C which can be added to $S_1 = (7,5,2,1)/(4,2,2,1)$ to form a border ribbon strip. The resulting border ribbon strips all have height 1.

- $(2) \subset (5,5) \subset (5,5,2)$ with height 1 and spin 3,
- $(2) \subset (5,3) \subset (5,5,2)$ with height 2 and spin 1.

Thus

$$\mathcal{X}_5^{\lambda/\mu}(q) = q^5 - 2q^3 + q.$$

4. The Murnaghan-Nakayama Rule

The core calculation of the paper [**Lam1**] (performed using the action of the Heisenberg algebra on the Fock space of $U_q(\widehat{\mathfrak{sl}}_n)$) is the ribbon Murnaghan-Nakayama Rule.

THEOREM 4.1 (Murnaghan-Nakayama Rule). *Let $k \geq 1$ be an integer and ν be a partition. Then*

$$\left(1 + q^{2k} + \cdots + q^{2k(n-1)}\right) p_k \mathcal{G}_\nu(X;q) = \sum_\mu \mathcal{X}_k^{\mu/\nu}(q) \mathcal{G}_\mu(X;q). \tag{4.1}$$

Also

$$k \frac{\partial}{\partial p_k} \mathcal{G}_\nu(X;q) = \sum_\mu \mathcal{X}_k^{\nu/\mu}(q) \mathcal{G}_\mu(X;q).$$

EXAMPLE 4.2. Let $n = 2$ and consider $(1+q^4)p_2 \cdot 1$. By the ribbon Murnaghan-Nakyama rule ($\mathcal{G}_0 = 1$), this should equal to

$$\mathcal{G}_{(4)} + q\mathcal{G}_{(3,1)} + (q^2-1)\mathcal{G}_{(2,2)} - q\mathcal{G}_{(2,1,1)} - q^2\mathcal{G}_{(1,1,1,1)}.$$

We can compute directly that

$$\mathcal{G}_{(4)} = h_2, \quad \mathcal{G}_{(3,1)} = qh_2, \quad \mathcal{G}_{(2,1,1)} = qe_2$$
$$\mathcal{G}_{(2,2)} = q^2 h_2 + e_2, \quad \mathcal{G}_{(1,1,1,1)} = q^2 e_2,$$

verifying Theorem 4.1 directly.

5. Murnaghan-Nakayama and Pieri

We now show that the 'ribbon Murnaghan-Nakayama' and 'ribbon Pieri' (to be made explicit in Section 6) rules are formally equivalent. In the case $n = 1$ we obtain a direct combinatorial proof that the usual Pieri and Murnaghan-Nakayama rules are equivalent.

LEMMA 5.1. *The power sum and homogeneous symmetric functions satisfy the following equation*

$$mh_m = p_{m-1}h_1 + p_{m-2}h_2 + \cdots + p_m.$$

PROOF. See (2.10) in [**Mac**]. ☐

Let V be a vector space over $\mathbb{C}(q)$ and v_λ be vectors in V labelled by partitions. Recall the definitions of $\mathcal{X}_k^{\mu/\lambda}(q)$, $\mathcal{K}_{\mu/\lambda,k}(q)$ and $\mathcal{L}_{\mu/\lambda,k}(q)$ from Section 3. Suppose $\{P_k\}$ are commuting linear operators satisfying

$$P_k v_\lambda = \sum_\mu \mathcal{X}_k^{\mu/\lambda}(q) v_\mu \quad \text{for all } k$$

then we will say that the Murnaghan-Nakayama rule holds.

Suppose $\{H_k\}$ are commuting linear operators on V satisfying

$$H_k v_\lambda = \sum_\mu \mathcal{K}_{\mu/\lambda,k}(q) v_\mu \quad \text{for all } k,$$

then we will say that Pieri formula holds.

Suppose $\{E_k\}$ are commuting linear operators on V satisfying

$$E_k v_\lambda = \sum_\mu \mathcal{L}_{\mu/\lambda,k}(q) v_\mu \quad \text{for all } k,$$

then we will say that dual-Pieri formula holds.

If the skew shapes μ/λ are replaced by λ/μ in the above formulae, we get adjoint versions of these formulae which can be thought of as lowering operators. Thus if a set of commuting linear operators $\left\{P_k^\perp\right\}$ satisfies

$$P_k^\perp v_\lambda = \sum_\mu \mathcal{X}_k^{\lambda/\mu}(q) v_\mu \quad \text{for all } k$$

then we will say the lowering Murnaghan-Nakayama rule holds, and similarly for $\left\{E_k^\perp\right\}$ and $\left\{H_k^\perp\right\}$.

Proposition 5.2. *Fix $n \geq 1$ as usual. Let $\{H_k\}$ and $\{P_k\}$ be commuting sets of linear operators satisfying the relations between h_k and p_k in Λ. Then the ribbon Murnaghan-Nakayama rule holds for $\{P_k\}$ if and only if the ribbon Pieri rule holds for $\{H_k\}$.*

(Sketch of Proof). The idea is to use Lemma 5.1 and to proceed by induction on k. Thus suppose that the Murnaghan-Nakayama rule holds for $\{P_k\}$ and the ribbon Pieri rule holds for H_i for $i \leq k$. Then writing

$$kH_k = H_{k-1}P_1 + \cdots + P_k$$

we see that the action of kH_k on v_λ can be described in terms of ordered pairs (S, T) consisting of a border ribbon strip S and horizontal ribbon strip T (such that S is added first to λ then T later).

For the case $n = 1$, an involution α can be defined on such pairs (S, T) which changes the sign of $(-1)^{h(S)}$. This involution α is given by

1. If the 'bottom' horizontal strip S_1 of S is such that $T \cup S_1$ is a horizontal strip then we set $\alpha(S, T) = (S - S_1, T \cup S_1)$
2. Otherwise T 'touches' S from below. Let $\alpha(S, T) = (S \cup T_1, T - T_1)$ where T_1 is the unique sub horizontal strip which can be attached to S to form another border strip.

In both cases the height of the border strip will change and one can check that this is an involution when it is well defined. The contributions of these strips to $kH_k v_\lambda$ cancel out since the total shape $\lambda \cup S \cup T$ is fixed. The involution fails to be defined in the situation that S and T are both horizontal strips such that $S \cup T$ is also a horizontal strip. This case gives exactly the contribution to kH_k, proving the inductive step.

The case for general n is more complicated, but the idea is similar. □

In fact we have the following theorem [**Lam1**].

Theorem 5.3. *Let $\{H_i\}$, $\{E_i\}$ and $\{P_i\}$ be commuting operators on a vector space V over $\mathbb{C}(q)$ satisfy the relations of h_i, e_i and p_i in Λ. Let v_λ be a set of vectors in V indexed by partitions. Suppose that one of the Pieri, dual-Pieri and Murnaghan-Nakayama holds, then all three holds. The same is true for the lowering operators satisfying the same relation.*

6. Ribbon Pieri Formulae

Let $n \geq 1$ be a fixed integer. Define the formal power series

$$H(t) = \prod_i \prod_{k=0}^{n-1} \frac{1}{1 - x_i q^{2k} t}$$

$$E(t) = \prod_i \prod_{k=0}^{n-1} \left(1 + x_i q^{2k} t\right).$$

As usual we may define symmetric functions $\mathbf{h}_k$ and $\mathbf{e}_k$ by $H(t) = \sum_k \mathbf{h}_k t^k$ and similarly for $\mathbf{e}_k$. Note that we have suppressed the integer n from the notation. We shall see later that the definitions of these power series are completely natural in the context of Robinson-Schensted ribbon insertion.

In plethystic notation, $\mathbf{h}_k = h_k[(1+q^2+\cdots+q^{(2n-2)})X]$ and $\mathbf{e}_k = e_k[(1+q^2+\cdots+q^{(2n-2)})X]$. The following theorem is an immediate consequence of Theorem 5.3 and Theorem 4.1.

Theorem 6.1 (Ribbon Pieri Rule). *Let λ be a partition. Then*

$$\mathbf{h}_k \mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_\mu(X;q) \tag{6.1}$$

where the sum is over all partitions μ such that μ/λ is a horizontal n-ribbon strip with k ribbons. Here $s(\mu/\lambda)$ refers to the spin of the unique tableau which is a horizontal ribbon strip of shape μ/λ. Also

$$\mathbf{e}_k \mathcal{G}_\lambda(X;q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_\mu(X;q)$$

where the sum is over all partitions μ such that μ/λ is a vertical n-ribbon strip with k ribbons. Here $s(\mu/\lambda)$ refers to the spin of the unique tableau which is a vertical ribbon strip of shape μ/λ.

Note that by Theorem 6.1, we have

$$\mathbf{h}_k = \sum_\lambda q^{\mathrm{mspin}(\lambda)} \mathcal{G}_\lambda(X;q)$$

where the sum is over all λ with no n-core such that $|\lambda| = kn$ with no more than n rows and $\mathrm{mspin}(\lambda)$ is the maximum spin of a ribbon tableau of shape λ. A similar formula holds for $\mathbf{e}_k$.

Example 6.2. Let $n = 3$, $k = 2$ and $\lambda = (3,1)$. Then

$$\mathbf{h}_2 \mathcal{G}_{(3,1)} = \mathcal{G}_{(9,1)} + q\mathcal{G}_{(6,2,2)} + q^2\mathcal{G}_{(4,4,2)} + q^2\mathcal{G}_{(6,1,1,1,1)} + q^3\mathcal{G}_{(3,3,2,1,1)} + q^4\mathcal{G}_{(3,2,2,2,1)}.$$

We should remark that dual-Pieri formulae also follows and is equivalent to a cospin branching formula of [**SSW**]. These dual formulae are in some sense easier as they essentially only rely on the fact that ribbon functions are symmetric.

7. The Ribbon Involution ω_n and the Ribbon Cauchy Identity

We now define an involution w_n on $\Lambda(q)$ which is essentially the involution $v \mapsto v'$ on the Fock space $\mathbf{F}$ of [**LT**]. However, this involution will turn out to be not just a semi-linear involution, but also a $\mathbb{C}$-algebra isomorphism of $\Lambda(q)$.

Definition 7.1. Define the *ribbon involution* $w_n : \Lambda(q) \to \Lambda(q)$ as the semi-linear map satisfying $w_n(q) = q^{-1}$ and

$$w_n(s_\lambda) = q^{(n-1)|\lambda|} s_{\lambda'}.$$

Theorem 7.2. *The map w_n is an $\mathbb{C}$-algebra homomorphism which is an involution. It maps $\mathcal{G}_{\lambda/\mu}$ into $\mathcal{G}_{(\lambda/\mu)'}$ for every skew shape λ/μ.*

The proof of the first statement is not difficult. The proof of the second statement requires the use of calculations in the Fock Space **F** which are generalisations of those in [**LT**], together with symmetric function manipulations.

Let us write the formal power series

$$\Omega(X;q) = \prod_{i,j} \prod_{k=0}^{n-1} \frac{1}{1 - x_i y_j q^{2k}}$$

$$\tilde{\Omega}(X;q) = \prod_{i,j} \prod_{k=0}^{n-1} \left(1 + x_i y_j q^{2k}\right).$$

Then we have:

Theorem 7.3 (Ribbon Cauchy Identity). *Fix n as usual and a n-core δ. Then*

$$\Omega(X;q) = \sum \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q)$$

and

$$\tilde{\Omega} = \sum_{\lambda \in \mathcal{P}_\delta} q^{(n-1)|\lambda/\tilde{\lambda}|} \mathcal{G}_{\lambda'}(X;q)\mathcal{G}_\lambda(Y;q^{-1}).$$

where the sum is over all λ such that $\tilde{\lambda} = \delta$.

Note that this does not imply that the $\mathcal{G}_\lambda$ form an orthonormal basis under a certain inner product, as they are not linearly independent.

(Sketch of Proof). Using results relating the Fock Space **F** and $\Lambda(q)$ in [**Lam1**] we have

$$s_\lambda[(1 + q^2 + \cdots + q^{2n-2})X] = \sum_\mu c^\lambda_\mu(q)\mathcal{G}_\mu(X;q)$$

where the sum is over all $\mu \in \mathcal{P}_\delta$. Now multiply both sides by $s_\lambda(Y)$ and sum over λ, giving the Cauchy identity. The dual Cauchy identity can be obtained via a calculation involving ω_n. ☐

The factor of $q^{(n-1)|\lambda/\tilde{\lambda}|}$ can be explained combinatorially by the fact that $s(T') = q^{(n-1)|\lambda/\tilde{\lambda}|} - s(T)$ for a ribbon tableau T and its conjugate T' satisfying $sh(T) = \lambda$.

8. Connections with Ribbon Insertion

In this section we put the ribbon Pieri formula (Theorem 6.1) and ribbon Cauchy identity (Theorem 7.3) in the context of ribbon Robinson-Schensted-Knuth (RSK) insertion, where both will be proven combinatorially and completely for the case $n = 2$.

8.1. Robinson-Schensted-Knuth for usual Young tableaux. Recall that the Robinson-Schensted bijection gives a bijection between permutations $w \in S_m$ and pairs of standard Young tableaux (see [**EC2**]):

$$w \mapsto (P(w), Q(w)).$$

The semistandard generalisation of this is a bijection between biwords w and pairs of semistandard tableaux $(P(w), Q(w))$ of the same shape. This immediately implies the usual Cauchy identity.

In fact the bijection is realised by the insertion algorithm which produces a semistandard tableau $T' = (T \leftarrow i)$ given a semistandard tableau T and a number i to insert. An *increasing insertion* property of Robinson-Schensted-Knuth insertion guarantees that $Q(w)$ will be semistandard. Let $i < j$. The increasing insertion property is the fact that the insertion path of i will always lie to the left of the path of j (if i is inserted before j). This property is crucial to a combinatorial proof (see [**EC2**, p. 341]) of the Pieri rule:

$$h_k s_\lambda = \sum_\mu s_\mu.$$

We may interpret h_k as the generating function for a k-tuple of increasing positive integers $(i_1 \leq i_2 \leq \cdots \leq i_k)$, and s_λ as the weight generating function of tableaux T with shape λ, as usual. Then a bijection from the left hand side to the right hand side is obtained by associating to a pair $((i_1, \cdots, i_k), T)$ the tableau

$$T' = ((\cdots((T \leftarrow i_1) \leftarrow i_2) \cdots) \leftarrow i_k).$$

The increasing insertion property guarantees that $sh(T')/\lambda$ is indeed a horizontal strip.

8.2. Domino insertion. The above discussion also leads to proofs for the domino $n = 2$ tableaux case. Barbasch and Vogan [**BV**] have defined domino insertion in connection with the primitive ideals of classical lie algebras. This was put into the usual bumping description by Garfinkle [**Gar**]. Recently, Shimozono and White [**SW**] have extended Garfinkle's description to the semistandard case and connected it with mixed insertion. They also observed that it had the crucial color-to-spin property. A straightforward extension to the non-empty 2-core case was presented in [**Lam**].

A colored biletter is an ordered triple (c, i, j) where $c \in \{0, 1\}$ is the color and $i, j \in \{1, 2, \ldots\}$. A colored biword ω is a multiset of colored biletters canonically ordered in some way, usually denoted in an array:

$$\mathbf{w} = \begin{pmatrix} c_1 & \cdots & c_m \\ i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}$$

THEOREM 8.1. *Fix a 2-core δ. There is a bijection between colored biwords $\mathbf{w}$ of length m with two colors $\{0, 1\}$ and pairs $(P_d(\mathbf{w}), Q_d(\mathbf{w}))$ of semistandard domino tableaux with the same shape $\lambda \in \mathcal{P}_\delta$ and $|\lambda| = 2m + |\delta|$ with the following properties:*

- *The bijection has the color-to-spin property:*

$$tc(\mathbf{w}) = s(P_d(\mathbf{w})) + s(Q_d(\mathbf{w})) \tag{8.1}$$

  where $tc(\mathbf{w})$ is the twice the sum of the colors in the top line of $\mathbf{w}$.
- *The weight of $P_d(\mathbf{w})$ is the weight of the lowest line of $\mathbf{w}$. The weight of $Q_d(\mathbf{w})$ is the weight of the middle line of $\mathbf{w}$.*

In fact the bijection is realized by an insertion procedure (denoted $(T \leftarrow \gamma_i)$ where T is a domino tableau and γ_i is either a horizontal or vertical domino labelled i) analogous to the usual Robinson-Schensted insertion.

This bijection immediately leads to the domino Cauchy identity ($n = 2$ in Theorem 7.3). In [**Lam**], we have also described two dual domino insertion algorithms which are bijections between 'dual colored biwords' and pairs of semistandard tableaux of conjugate shape. This proves the dual domino Cauchy formula.

It further turns out that domino insertion has the following domino increasing insertion property. This was first shown by Shimozono and White by connecting domino insertion with mixed insertion. [**Lam**] gives a different proof using growth diagrams. This domino increasing insertion property can be described by specifying a total order $<$ on dominoes as follows (γ_i denotes a domino labelled i)

1. If γ_i is horizontal and γ_j vertical then $\gamma_i > \gamma_j$.
2. If γ_i and γ_j are both horizontal then $\gamma_i > \gamma_j$ if and only if $i > j$.
3. If γ_i and γ_j are both vertical then $\gamma_i > \gamma_j$ if and only if $i < j$.

Under this order, domino insertion also has a increasing insertion property,

LEMMA 8.2. *Let T be a domino tableau without the labels i and j. Set $T' = (T \leftarrow \gamma_i)$ and $T'' = (T' \leftarrow \gamma_j)$ for some dominoes γ_i and γ_j. Then $sh(T'/T)$ lies to the left of $sh(T''/T')$ if and only if $\gamma_i < \gamma_j$.*

Similarly, the dual domino insertion has a property which is dual to this. This increasing property is retained when the bijection is extended to the semistandard case (see [**SW, Lam**] for details).

Proposition 8.3. *Semistandard domino insertion gives a combinatorial proof of the Pieri rule (Theorem 6.1) for $n = 2$. Dual semistandard domino insertion gives a combinatorial proof of the dual Pieri rule for $n = 2$.*

Proof. From the formal power series $H(t)$, it is easy to see that $\mathbf{h}_k$ is the weight generating function for multisets $\Gamma = \{\gamma_i\}_{i=1}^k$ of labelled dominoes of size k, where the weight of a labelled domino γ_i is given by

$$w(\gamma_i) = q^{2s(\gamma_i)} x_i.$$

Now fix a shape λ. Let S_1 be the set of pairs (Γ, T) where Γ is a multiset of dominoes of size k and T is a semistandard domino tableau of shape λ. Let S_2 be the set of semistandard tableaux T' such that $sh(T')/\lambda$ is a horizontal domino strip of size k. We define a map $\alpha : S_1 \to S_2$ by

$$\alpha(\Gamma, T) = ((\cdots((T \leftarrow \gamma_1)) \leftarrow \gamma_2) \cdots) \leftarrow \gamma_k),$$

where γ_i runs over the dominoes within Γ. Here the dominoes are inserted in the order of the increasing insertion property described above ensuring that the change in shape $sh(T')/sh(T)$ is a horizontal strip. Taking the weights of these sets and using the color-to-spin property of domino insertion we see obtain Theorem 6.1 for $n = 2$. Using Theorem 8.1 one sees that α is a bijection. The proof for the dual case is exactly analagous. □

8.3. Shimozono and White's ribbon insertion. Shimozono and White [**SW2**] have described a ribbon insertion algorithm for general n. This can be described in a traditional bumping fashion or in terms of Fomin's growth diagrams [**Fom1, Fom2**].

The ribbon insertion algorithm of [**SW2**] has the usual weight preserving properties, but also the spin to color property (8.1) which an earlier ribbon-RSK algorithm of Stanton and White [**SW1**] did not have. However, the algorithm stops short of being a bijection between colored biwords (with n colors) and pairs of semistandard ribbon tableaux. The algorithm is only described as a bijection π between colored words $\mathbf{w}$ (not biwords) and a pair $(P_r(\mathbf{w}), Q_r(\mathbf{w}))$ where $P_r(\mathbf{w})$ is a semistandard ribbon tableau and $Q_r(\mathbf{w})$ is a standard ribbon tableau. In particular the Cauchy identity of Theorem 7.3 does not immediately follow. The algorithm also does not seem to possess a *ribbon increasing insertion* property. However one can at least salvage the following, which is just the first Pieri rule.

Proposition 8.4. *Shimozono and White's bijection π gives a combinatorial proof that*

$$(1 + q^2 + \ldots + q^{2(n-1)}) h_1 \mathcal{G}_\lambda(X; q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_\mu(X; q)$$

where the sum is over all μ such that μ/λ is a n-ribbon.

Proof. As before we construct a weight preserving bijection between the two sides of the Pieri rule by:

$$(T, (c, j)) \mapsto T' = (T \leftarrow (c, j)).$$

Here (c, j) denotes an n-ribbon with color (or spin) c and label j. The color c ranges from 0 to $n-1$ and h_1 is just the generating function for the labels j. □

Shimozono and White's ribbon insertion is determined by forcing all ribbons to bump by rows to another ribbon of the same spin (at least in the standard case). It is possible however to insist that all ribbons of a particular spin bump by columns instead. Unfortunately, it appears that none of these algorithms have a ribbon increasing insertion property.

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