RIBBON TABLEAUX AND THE HEISENBERG ALGEBRA

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Abstract. In [LLT] Lascoux, Leclerc and Thibon introduced symmetric functions $\mathcal{G}_\lambda$ which are spin and weight generating functions for ribbon tableaux. This article is aimed at studying these functions in analogy with Schur functions. In particular we will describe:

- a Pieri and dual-Pieri formula for ribbon functions,
- a ribbon Murnaghan-Nakayama formula,
- ribbon Cauchy and dual Cauchy identities,
- and a $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ which sends each $\mathcal{G}_\lambda$ to $\mathcal{G}_{\lambda'}$.

Our study of the functions $\mathcal{G}_\lambda$ will be connected to the Fock space representation $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ via a linear map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ which sends the standard basis of $\mathbf{F}$ to the ribbon functions. Kashiwara, Miwa and Stern [KMS] have shown that a copy of the Heisenberg algebra H acts on $\mathbf{F}$ commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$. Identifying the Fock Space of H with the ring of symmetric functions $\Lambda(q)$ we will show that $\mathbf{\Phi}$ is in fact a map of H-modules with remarkable properties. The study of this map will lead to our identities concerning ribbon tableaux generating functions. We will also give a combinatorial proof that the ribbon Murnaghan-Nakayama and Pieri rules are formally equivalent.

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1. Introduction

Let $n \geq 1$ be a fixed integer and λ a partition with empty n-core. In analogy with the combinatorial definition of the Schur functions, Lascoux, Leclerc and Thibon [LLT] have defined a family of symmetric functions $\mathcal{G}_\lambda^{(n)}(X;q) \in \Lambda(q)$ by:

$$\mathcal{G}_\lambda^{(n)}(X;q) = \sum_T q^{s(T)} x^{w(T)}$$

where the sum is over all *semistandard n-ribbon tableaux* (see Figure 1) of shape λ, and $s(T)$ and $w(T)$ are the spin and weight of T respectively. The definition of a semistandard ribbon tableau is analagous to the definition of a semistandard Young tableau, with boxes replaced by ribbons (or border strips) of length n. We shall loosely call the functions $\mathcal{G}_\lambda(X;q)$ *ribbon functions* and suppress the notation for n when no confusion occurs.

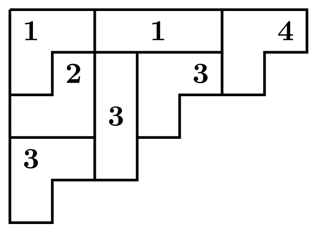

Figure 1. A semistandard 3-ribbon tableau with shape $(7,6,4,3,1)$, weight $(2,1,3,1)$ and spin 7.

When $q = 1$ the ribbon functions become products of usual Schur functions. However, when the parameter q is introduced, it is no longer obvious that the functions $\mathcal{G}_\lambda(X;q)$ are symmetric. The main aim of this paper will be to develop the theory of ribbon functions in the same way Schur functions are studied in the ring of symmetric functions. Our main results are:

- A ribbon Pieri formula (Theorem 11):

$$h_k[\left(1 + q^2 + \cdots + q^{2(n-1)}\right) X]\mathcal{G}_\nu(X;q) = \sum_\mu q^{s(\mu/\nu)}\mathcal{G}_\mu(X;q).$$

  where the sum is over all μ such that μ/ν is a horizontal ribbon strip of size k. The notation $h_k[\left(1 + q^2 + \cdots + q^{2(n-1)}\right) X]$ denotes a plethysm.
- A ribbon Murnaghan-Nakayama-rule (Theorem 20):

$$\left(1 + q^{2k} + \cdots + q^{2k(n-1)}\right) p_k \mathcal{G}_\nu(X;q) = \sum_\mu \mathcal{X}^k_{\mu/\nu}(q)\mathcal{G}_\mu(X;q).$$

  where $\mathcal{X}^k_{\mu/\nu}(q)$ can be expressed as an alternating sum of spins over certain 'border n-ribbon strips' of size k.
- A ribbon Cauchy (and dual Cauchy) identity (Theorem 26):

$$\sum_\lambda \mathcal{G}_{\lambda/\delta}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q) = \prod_{i,j}\prod_{k=0}^{n-1} \frac{1}{1 - x_i y_j q^{2k}}$$

  where the sum is over all partitions λ with a fixed n-core δ. A combinatorial proof of this was given recently by van Leeuwen [vL].

- A $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ (Theorem 24) satisfying
$$\omega_n(\mathcal{G}_\lambda(X;q)) = \mathcal{G}_{\lambda'}(X;q).$$
Even the existence of a linear map with such a property is not obvious as the functions $\mathcal{G}_\lambda$ are not linearly independent.

The study of ribbon functions has been focused on the q-Littlewood Richardson coefficients $c^\mu_\lambda(q)$ of the expansion of $\mathcal{G}_\lambda(X;q)$ in the Schur basis:
$$\mathcal{G}_\lambda(X;q) = \sum_\mu c^\mu_\lambda(q) s_\lambda(X).$$
These are q-analogues of Littlewood Richardson coefficients. Leclerc and Thibon [LT] have shown that these are coefficients of global bases of the Fock Space $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$. Results of Varagnolo and Vasserot [VV] then imply that they are parabolic Kazhdan-Lusztig polynomials of type A. Finally, geometric results of Kashiwara and Tanisaki [KT] show that they are polynomials in q with non-negative coefficients. Much interest has also developed in connecting ribbon tableaux and the q-Littlewood Richardson coefficients to rigged configurations and the generalised Kostka polynomials defined by Kirillov and Shimozono [KS], Shimozono and Weyman [SW] and Schilling and Warnaar [SchW].

In a mysterious and recent development, Haglund et. al. [HHLRU] have conjectured connections between diagonal harmonics and ribbon functions. More recently, Haglund, Haiman and Loehr [HHL] have found an expression for Macdonald polynomials in terms of skew ribbon functions.

To prove that the functions $\mathcal{G}^{(n)}_\lambda(X;q)$ were symmetric Lascoux, Leclerc and Thibon connected them to the (level 1) Fock space representation $\mathbf{F}$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. The crucial property of $\mathbf{F}$ is that it affords an action of a Heisenberg algebra H, commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$, discovered by Kashiwara, Miwa and Stern [KMS]. In particular, they showed that as a $U_q(\widehat{\mathfrak{sl}}_n) \times H$-module, $\mathbf{F}$ decomposes as
$$\mathbf{F} \cong V_{\Lambda_0} \otimes \mathbb{C}(q)[H_-]$$
where V_{Λ_0} is the highest weight representation of $U_q(\widehat{\mathfrak{sl}}_n)$ with highest weight Λ_0 and $\mathbb{C}(q)[H_-]$ is the usual Fock space representation of the Heisenberg algebra. Our results imply a description of the action of the bosonic operators $B_k \in H$ on $\mathbf{F}$ in terms of 'border ribbon strips'.

Whereas the symmetry of the functions $\mathcal{G}_\lambda(X;q)$ relies only on the action of the lower half of the Heisenberg algebra, the Pieri and Cauchy identities will follow from the full action of the Heisenberg algebra. In fact, there is a formal relationship between the existence of Pieri and Cauchy identities and actions of a Heisenberg algebra, which will be the subject of a later paper. We will not go into such generality here, as there are many details particular to the ribbon function case.

The connection between ribbon functions and the action of the Heisenberg algebra is made explicit by showing (Theorem 9) that the map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ defined by
$$|\lambda\rangle \longmapsto \mathcal{G}_\lambda$$
is a map of H-modules, after identifying $\mathbb{C}(q)[H_-]$ with the ring of symmetric functions $\Lambda(q)$ in the usual way. The map $\mathbf{\Phi}$ has the further remarkable property that it changes certain linear maps into algebra maps, as follows.

Lascoux, Leclerc and Thibon [LLT1] have constructed a global basis of $\mathbf{F}$ which extends Kashiwara's global crystal basis of V_{Λ_0}. They used a bar involution $^- : \mathbf{F} \to \mathbf{F}$ which extends Kashiwara's involution on V_{Λ_0}. Another semi-linear involution, denoted $v \mapsto v'$ was also introduced and further studied in [LT] which satisfied the property $\langle \overline{u}, v\rangle = \langle u', \overline{v'}\rangle$ for $u, v \in \mathbf{F}$ and $\langle |\lambda\rangle, |\mu\rangle\rangle = \delta_{\lambda\mu}$ the standard inner product on $\mathbf{F}$. We shall see that if we restrict $\mathbf{\Phi}$ to the space of highest weight vectors of $\mathbf{F}$ for the $U_q(\widehat{\mathfrak{sl}}_n)$ action, then both involutions become algebra isomorphisms under the map $\mathbf{\Phi}$. In particular the 'image' of the involution $v \mapsto v'$ is simply ω_n.

Organisation. In Section 2, we begin by defining ribbon tableaux and in Section 3 we review some notation in symmetric function theory. In Section 4, we define the main objects of the paper, the ribbon functions. In Section 5 we introduce the Fock space $\mathbf{F}$ and certain operators on $\mathbf{F}$ defined in terms of ribbons. In Section 6, we describe, following [KMS], an action of the Heisenberg algebra on $\mathbf{F}$. We connect the action of the Heisenberg algebra to ribbon functions in Section 7. Section 8 contains the main representation theoretic result of the paper: a map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ which is a map of modules over the Heisenberg algebra. In Section 9, we describe the ribbon Pieri rule. In Section 10 we define a new combinatorial object called a border ribbon strip and in Section 11 we prove that the ribbon Pieri rule formally implies a ribbon Murnaghan-Nakayama rule which involves border ribbon strips. This is connected to ribbon functions and the action of the bosonic operators on the Fock space in Section 12. In Sections 13 and 14, we study an involution $v \mapsto v'$ on $\mathbf{F}$ and its 'image' ω_n in $\Lambda(q)$. In Section 15, we prove the ribbon Cauchy identity, and extend this to skew Schur functions in Section 16 where we also define *super ribbon functions*. In Section 17, we study the 'images' in $\Lambda(q)$ of the bar involution and inner products of $\mathbf{F}$.

Acknowledgements. This work is part of my dissertation written under the guidance of Richard Stanley. I am indebted to him for suggesting the study of ribbon tableaux and for providing me with assistance throughout. I thank Marc van Leeuwen for comments on an earlier version of this paper. I also thank Mark Shimozono and Ole Warnaar for pointing out certain references.

2. Ribbon tableaux

A distinguished integer $n \geq 1$ will be fixed throughout the whole paper. When $n = 1$, the reader may check that we recover the classical theory of Young tableaux and Schur functions. For general notation in this section we refer the reader to [Mac], and for ribbon tableaux in particular to [LLT].

A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a list of non-increasing integers. We will call l the *length* of λ, and denote it by $l(\lambda)$. We will say that λ is a partition of $\lambda_1 + \lambda_2 + \ldots + \lambda_l = |\lambda|$ and write $\lambda \vdash |\lambda|$. A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is an ordered list of non-negative integers. As above, we will say that α is a composition of $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. We use the usual notation concerning partitions and do not distinguish between a partition and its Young diagram. Let $m_k(\lambda)$ denote the number of parts of λ equal to k and let λ' denote the *conjugate* of λ.

A skew shape λ/μ is a *horizontal strip* if it contains at most one square in each column. A skew shape λ/μ is a *border strip* if it is connected, and does not contain any 2×2 square. The *height* $h(b) \in \mathbb{N}$ of a border strip b is the number of rows in it, minus 1. A *border strip tableau* is a chain of partitions

$$\mu^0 \subset \mu^1 \subset \cdots \subset \mu^r$$

such that each μ^{i+1}/μ^i is a border strip. The height of a *border strip tableau* T is the sum of the heights of its border strips. When a border strip has n squares for the distinguished integer n, we will call it a *n-ribbon* or just a *ribbon*. The height of the ribbon r will then be called its *spin* $s(r)$. The reader should be cautioned that in the literature the spin is often defined as half of this.

A *semistandard* tableau of shape λ/μ is a filling of each square $(i, j) \in \lambda/\mu$ with a positive integer such that the rows are non-decreasing and the columns are increasing. The weight $w(T)$ of such a tableau T is the composition α such that α_i is the number of occurrences of i in T. The tableau is *standard* if the numbers which occur are exactly those of $[m] = \{1, 2, \ldots, m\}$ for some integer m.

Let λ be a partition. Its *n-core* is denoted $\tilde{\lambda}$. The *n-quotient* of λ is denoted by $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$. We shall write $\mathcal{P}$ for the set of partitions. Let $\mathcal{P}_\delta$ to denote the set of partitions λ such that $\tilde{\lambda} = \delta$ for an n-core $\delta = \tilde{\delta}$. A *ribbon tableau* T of shape λ/μ is a tiling of λ/μ by n-ribbons and a filling of each ribbon with a positive integer (see Figure 1). If these numbers are exactly those of $[m]$, for some m, then the tableau is called *standard*. We will use the convention that a ribbon tableau of shape λ where $\tilde{\lambda} \neq \emptyset$ is simply a ribbon tableau of shape $\lambda/\tilde{\lambda}$. A ribbon tableau is *semistandard* if for each i

(1) removing all ribbons labelled j for $j > i$ gives a valid skew shape $\lambda_{\leq i}/\mu$ and,
(2) the subtableau containing only the ribbons labelled i form a *horizontal n-ribbon strip*.

A tiling of a skew shape λ/μ by n-ribbons is a *horizontal ribbon strip* if the topright-most square of every ribbon touches the northern edge of the shape (see Figure 2). Abusing notation, we will also call the skew shape λ/μ a horizontal ribbon strip λ/μ if such a tiling exists (which is necessarily unique). Similarly, one has the notion of a *vertical ribbon strip*.

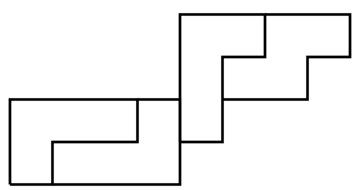

FIGURE 2. A horizontal 4-ribbon strip with spin 5.

We will often think of a ribbon tableau as a chain of partitions

$$\tilde{\lambda} = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

where each μ^{i+1}/μ^i is a horizontal ribbon strip. The *spin* $s(T)$ of a ribbon tableau T is the sum of the spins of its ribbons. If λ/μ is a horizontal ribbon strip then $s(\lambda/\mu)$ denotes the spin of the unique tiling of λ/μ such that the topright-most square of every ribbon touches the northern edge of the shape. The *weight* $w(T)$ of a tableau is the composition counting the occurrences of each value in T.

Littlewood's *n-quotient map* ([Lit], see also [StW]) gives a weight preserving bijection between semistandard ribbon tableaux T of shape λ and n-tuples of semistandard Young tableau $\{T^{(0)}, \ldots, T^{(n-1)}\}$ of shapes $\lambda^{(i)}$ respectively. Abusing language, we shall also refer to $\{T^{(0)}, \ldots, T^{(n-1)}\}$ as the n-quotient of T. Schilling, Shimozono and White [SSW] and separately Haglund et. al. [HHLRU] have described the spin of a ribbon tableau in terms of an inversion number of the n-quotient. None of our proofs will require the use of the

n-quotient but occasionally we will comment on the $q = 1$ case for which the n-quotient will be important.

The n-quotient map can be described as follows for the special case where the n-core is empty. A *diagonal* diag_d of a shape λ consists of all squares (i, j) such that $i - j = d$. If we draw all diagonals of the form diag_{dn} then each ribbon will intersect each such diagonal exactly once (in a single cell). A ribbon's squares are linearly ordered from top right to bottom left. Suppose the diagonal diag_{dn} intersects a ribbon r at the k^{th} square from the top right. Then the ribbon r is sent under the n-quotient map to a square in the diagonal diag_d of $\lambda^{(k)}$. The numbers in the ribbon tableau of Figure 1 have been placed along the diagonals diag_{dn}. Figure 3 shows its 3-quotient.

$$T^{(0)} = \begin{array}{|c|c|c|}\hline \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \end{array} \qquad T^{(1)} = \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{1} \\ \hline \mathbf{3} & \mathbf{3} \\ \hline \end{array} \qquad T^{(2)} = \emptyset$$

FIGURE 3. The 3-quotient of the ribbon tableaux T of Figure 1.

3. Symmetric functions

We review some standard notation in symmetric function theory (see [Mac] for details).

Let $\Lambda = \Lambda_{\mathbb{C}}$ denote the *ring of symmetric functions* over $\mathbb{C}$. We will write $\Lambda(q)$ for $\Lambda_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}(q)$. It is well known that the *Schur functions* s_λ are orthogonal with respect to the *Hall inner product* $\langle \, , \, \rangle$ on Λ. We will denote the *homogeneous, elementary, monomial* and *power sum* symmetric functions by h_λ, e_λ, m_λ and p_λ respectively. Recall that we have $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ where $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. Each of $\{p_i\}$, $\{e_i\}$ and $\{h_i\}$ generate Λ. We will write X to mean $(x_1, x_2, \ldots)$. Thus $s_\lambda(X) = s_\lambda(x_1, x_2, \ldots)$.

Recall that the *Kostka numbers* $K_{\lambda\mu}$ are defined by $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$. We will denote the inverse Kostka numbers by $\kappa_{\lambda\mu}$, so that $m_\mu = \sum_\lambda \kappa_{\lambda\mu} s_\lambda$.

Let $f \in \Lambda$. We recall the definition of the *plethysm* $g \mapsto g[f]$. Write $g = \sum_\lambda c_\lambda p_\lambda$. Then we have

$$g[f] = \sum_\lambda c_\lambda \prod_{i=1}^{l(\lambda)} f(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots).$$

Thus the plethysm by f is the (unique) algebra endomorphism of Λ which sends $p_k \mapsto f(x_1^k, x_2^k, \ldots)$. When $f(x_1, x_2, \ldots; q) \in \Lambda(q)$ for a distinguished element q, we define the plethysm as $p_k \mapsto f(x_1^k, x_2^k, \ldots; q^k)$. Note that plethysm does not commute with specialising q to a complex number.

For example, the plethysm by $(1+q)p_1$ is given by sending $p_k \mapsto (1+q^k)p_k$ and extending to an algebra isomorphism $\Lambda(q) \to \Lambda(q)$. In such situations we will write $f[(1+q)X]$ for $f[(1+q)p_1]$.

We are mainly concerned with the plethysm given by $(1 + q^2 + \cdots + q^{2(n-1)})p_1$. We will use $\Upsilon_{q,n}$ to denote the map $\Lambda(q) \to \Lambda(q)$ given by $f \mapsto f[(1 + q^2 + \cdots + q^{2(n-1)})X]$. Note that $p_k[(1 + q^2 + \cdots + q^{2(n-1)})X] = (1 + q^{2k} + \cdots + q^{2k(n-1)})p_k(X)$.

4. Lascoux, Leclerc and Thibon's ribbon functions

We now define the central objects of this paper as introduced by Lascoux, Leclerc and Thibon. The integer n is fixed throughout and suppressed in the notation.

Definition 1 ([LLT])**.** Let λ/μ be a skew partition, tileable by n-ribbons. Define the symmetric functions $\mathcal{G}_{\lambda/\mu}(X;q) \in \Lambda(q)$ as:

$$\mathcal{G}_{\lambda/\mu}(X;q) = \sum_T q^{s(T)} x^{w(T)}$$

where the sum is over all semistandard ribbon tableaux T of shape λ/μ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. These functions will be loosely called *ribbon functions*.

When $\mu = \emptyset$ we will write $\mathcal{G}_\lambda(X;q)$ in place of $\mathcal{G}_{\lambda/\emptyset}(X;q)$. The fact that the functions $\mathcal{G}_{\lambda/\mu}(X;q)$ are symmetric is not obvious from the combinatorial definition. However, using the action of the Heisenberg algebra on the Fock space $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$, the proof is immediate ([LLT]) and given in Theorem 6.

Let λ/μ be a skew shape tileable by n-ribbons. Then define

$$\mathcal{K}_{\lambda/\mu,\alpha}(q) = \sum_T q^{s(T)},$$

the spin generating function of all semistandard ribbon tableaux T of shape λ/μ and weight α. Thus $\mathcal{G}_{\lambda/\mu}(X;q) = \sum_\alpha \mathcal{K}_{\lambda/\mu,\alpha}(q) x^\alpha$.

When $q = 1$, the ribbon functions become products of Schur functions (see [LLT]):

$$\mathcal{G}_\lambda(X;1) = s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}. \tag{1}$$

This is a consequence of Littlewood's n-quotient map. In fact, up to sign, $\mathcal{G}_\lambda(X;1)$ is essentially $\phi_n(s_\lambda)$ where ϕ_n is the adjoint operator to taking the plethysm by p_n ([LLT]). More generally, $\mathcal{G}_{\lambda/\mu}(X;q)$ reduces to a product of skew Schur functions at $q = 1$.

Remark 1. In [LLT], another set of symmetric functions $\mathcal{H}_\lambda(X;q)$ defined by $\mathcal{H}_\lambda(X;q) = \mathcal{G}_{n\lambda}(X;q)$ is studied. It is not hard to see that $\mathcal{H}_\lambda(X;1) = s_\lambda(X) + \sum_{\mu \prec \lambda} d_{\lambda,\mu} s_\mu(X)$ for some $d_{\lambda,\mu} \in \mathbb{Z}$ where $\prec$ denotes the usual dominance order on partitions. Thus the functions $\mathcal{H}_\lambda(X;q)$ form a basis of $\Lambda(q)$ over $\mathbb{C}(q)$. In [LLT] it is shown that the 'cospin' version $\tilde{\mathcal{H}}_\lambda(X;q)$ generalise the modified Hall-Littlewood functions $Q'(X;q)$.

5. Two representations of Λ on the Fock space

Let $\mathbf{F}$ denote the vector space over $\mathbb{C}(q)$ spanned by a countable basis $|\lambda\rangle$ indexed by $\lambda \in \mathcal{P}$. We will call $\mathbf{F}$ the *Fock Space*. The Fock space is equipped with a natural inner product $\langle .,. \rangle$ satisfying $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$ (for notational clarity we sometimes only write the partition λ inside the inner product instead of $|\lambda\rangle$). Note that our notation for $\mathbf{F}$ from here onwards differs with that of the literature by the change of variables $q \mapsto -q^{-1}$.

Following [LLT], we define linear operators $\mathcal{V}_k \in \mathrm{End}(\mathbf{F})$ for $k \geq 1$ on the Fock Space $\mathbf{F}$, in terms of ribbon tableaux. Define $\mathcal{V}_k$ by

$$\mathcal{V}_k |\lambda\rangle = \sum_\mu q^{s(\mu/\lambda)} |\mu\rangle,$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of size k. The following result is due to [LLT], relying on a result of [KMS] that we will give shortly (Theorem 5).

Proposition 2 ([LLT])**.** *The operators $\mathcal{V}_k$ commute.*

Define a representation $\phi : \Lambda \to \mathrm{End}(\mathbf{F})$ of the symmetric functions on the Fock Space by

$$\phi : h_k \longmapsto \mathcal{V}_k.$$

By Proposition 2 and the fact that $\{h_k\}$ generate Λ, this definition extends to a representation of Λ. Now let $\psi : \Lambda \to \mathrm{End}(\mathbf{F})$ be the adjoint representation of Λ on $\mathbf{F}$, with respect to the inner product $\langle .,. \rangle$. Thus $\mathcal{U}_k = \psi(h_k)$ acts on a basis element $|\lambda\rangle$ by

$$\mathcal{U}_k|\lambda\rangle = \sum_{\nu} q^{s(\lambda/\nu)}|\nu\rangle,$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of size k. So $\mathcal{V}_k$ adds horizontal ribbon strips while $\mathcal{U}_k$ removes horizontal ribbon strips. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, we let $\mathcal{V}_\alpha$ denote the operator $\mathcal{V}_{\alpha_l} \cdots \mathcal{V}_{\alpha_2}\mathcal{V}_{\alpha_1}$ and similarly for $\mathcal{U}_\alpha$.

These representations of Λ are graded in the following way. If $\deg(|\lambda\rangle) = |\lambda|$ then $\phi(f)$ has degree $n \cdot \deg(f)$ for a homogeneous symmetric function f with degree $\deg(f)$.

Set $\tilde{\mathcal{V}}_k = \phi(e_k)$ and $\tilde{\mathcal{U}}_k = \psi(e_k)$. The following result is due to Leclerc and Thibon.

Theorem 3 ([LT]). *The operator $\tilde{\mathcal{V}}_k$ acts on $\mathbf{F}$ by*

$$\tilde{\mathcal{V}}_k|\lambda\rangle = \sum_{\mu} q^{s(\mu/\lambda)}|\mu\rangle,$$

where the sum is over all μ such that μ/λ is a vertical n-ribbon strip of size k. Similarly, $\tilde{\mathcal{U}}_k$ acts by

$$\tilde{\mathcal{U}}_k|\lambda\rangle = \sum_{\nu} q^{s(\lambda/\nu)}|\nu\rangle,$$

where the sum is over all ν such that λ/ν is a vertical n-ribbon strip of size k.

For convenience we shall define $S_\lambda = \phi(s_\lambda)$.

6. The Heisenberg algebra

The *Heisenberg Algebra* H is the associative algebra with 1 generated over $\mathbb{C}(q)$ by a countable set of generators $\{B_k : k \in \mathbb{Z} - \{0\}\}$ satisfying

$$[B_k, B_l] = l \cdot a_l(q) \cdot \delta_{k,-l} \tag{2}$$

for some elements $a_l(q) \in \mathbb{C}(q)$ satisfying $a_l(q) = a_{-l}(q)$. (Often the element 1 is called the central element and denoted c, but we will not need this generality). The *Fock space representation* $\mathbb{C}(q)[H_-]$ of H is the polynomial algebra

$$\mathbb{C}(q)[H_-] \cong \mathbb{C}(q)[B_{-1}, B_{-2}, \ldots].$$

The elements B_{-k} for $k \geq 1$ act by multiplication on $\mathbb{C}(q)[H_-]$. The action of B_k for $k \geq 1$ is given by (2) and the relation $B_k \cdot 1 = 0$ for $k \geq 1$.

An explicit construction of $\mathbb{C}(q)[H_-]$ is given by $\Lambda(q)$. We may identify B_k as the following operators:

$$B_k \longmapsto \begin{cases} f \longmapsto a_{-k}(q)p_{-k} \cdot f & \text{for } k < 0 \\ f \longmapsto k\frac{\partial}{\partial p_k} f & \text{for } k > 0. \end{cases}$$

Under this identification, the operators B_k have degree $-k$.

A standard lemma that we shall need later is

Lemma 4. *Let $k \geq 1$ be an integer and λ be a partition. Then*

$$B_k B_{-\lambda} = k a_k(q) m_k(\lambda) B_{-\mu} + B_{-\lambda} B_k$$

where $m_k(\lambda)$ is the number of parts of λ equal to k and μ is λ with one less part equal to k. If $m_k(\lambda) = 0$ to begin with then the first term is just 0.

Proof. We may commute B_k with $B_{-\lambda_i}$ immediately for parts $\lambda_i \neq k$. For each part equal to k, using the relation $[B_{-k}, B_k] = k a_k(q)$ introduces one term of the form $k a_k(q) B_{-\mu}$. □

The following theorem is due to Kashiwara, Miwa and Stern [KMS], though the connection with ribbon tableaux was first established in [LLT].

Theorem 5. *The algebra $\mathcal{A} \subset \mathrm{End}(\mathbf{F})$ generated by the two algebras $\phi(\Lambda)$ and $\psi(\Lambda)$ is isomorphic to a copy of the Heisenberg algebra H with parameters $a_l(q) = 1 + q^{2l} + \cdots + q^{2(n-1)l}$ for $l > 0$. The isomorphism $\vartheta : H \to \mathcal{A}$ is given by*

$$\vartheta : B_k \longmapsto \begin{cases} \phi(p_{-k}) & \text{if } k < 0 \\ \psi(p_k) & \text{if } k > 0. \end{cases}$$

Thus we have

$$[\phi(p_k), \psi(p_l)] = k \frac{1 - q^{2nk}}{1 - q^{2k}} \delta_{k,l}.$$

We call this representation of the Heisenberg algebra on the Fock space $\Theta : H \to \mathrm{End}(\mathbf{F})$. The operators $\phi(p_k)$ and $\psi(p_k)$ are known as *bosonic operators*. From now on, the Heisenberg algebra will always refer to the algebra with parameters $a_l(q) = 1 + q^{2l} + \cdots + q^{2(n-1)l}$ for $l > 0$.

For later use, we also define $\mathcal{X}^k_{\lambda/\mu}(q) \in \mathbb{C}[q]$ by $B_{-k}|\mu\rangle = \sum_\lambda \mathcal{X}^k_{\lambda/\mu}(q)|\lambda\rangle$ for $k > 0$. Since B_k is adjoint to B_{-k} with respect to $\langle .,. \rangle$, we also have $B_k|\lambda\rangle = \sum_\lambda \mathcal{X}^k_{\lambda/\mu}(q)|\mu\rangle$ for $k > 0$. We will show in Section 12 that the coefficients $\mathcal{X}^k_{\lambda/\mu}(q)$ can be described in terms of 'border ribbon strips'.

An elementary proof of Theorem 5 using the combinatorics of ribbons will appear in [Lam1].

7. Connection with ribbon functions and q-Littlewood Richardson coefficients

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a composition and λ/μ a skew shape tileable by n-ribbons. By definition, we have $[x^\alpha]\mathcal{G}_{\lambda/\mu}(X;q) = \langle \mathcal{V}_\alpha|\mu\rangle, |\lambda\rangle\rangle$, where $[x^\alpha]f$ denotes the coefficient of the monomial x^α in a power series f. The following theorem is immediate from Proposition 2.

Theorem 6 ([LLT])**.** *The generating functions $\mathcal{G}_{\lambda/\mu}(X;q)$ are symmetric functions.*

Proof. If β is a rearrangement of the parts of α then $\mathcal{V}_\beta = \mathcal{V}_\alpha$ by Proposition 2 so that $[x^\alpha]\mathcal{G}_{\lambda/\mu}(X;q) = [x^\beta]\mathcal{G}_{\lambda/\mu}(X;q)$. □

An elementary (but still not completely combinatorial) proof of Theorem 6 will appear in [Lam2]. Whereas the commutation of the operators $\mathcal{V}_k$ leads to the symmetry of the ribbon functions, we shall see later that the full Heisenberg algebra action gives rise to "ribbon" Pieri and Cauchy identities.

By Theorem 6, we may write

$$\mathcal{G}_{\lambda/\mu}(X;q) = \sum_{\rho} c^{\rho}_{\lambda/\mu}(q) s_\rho(X)$$

for some polynomials $c^{\rho}_{\lambda/\mu}(q)$ known as *q-Littlewood-Richardson coefficients* [LLT]. Leclerc and Thibon [LT] have shown that $c^{\rho}_{\lambda}(q) \in \mathbb{N}[q]$ for the case $\mu = \emptyset$. A combinatorial description of these coefficients is also known for $n = 2$ and is due to Carré and Leclerc [CL]. When λ has empty n-core, the q-Littlewood Richardson coefficients are usually written in terms of the n-quotient $c^{\nu}_{\lambda}(q) = c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}(q)$ and by (1) we have $c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}(1) = c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}$, a classical Littlewood-Richardson coefficient.

The following lemma is a slight generalisation of a result in [LT].

Lemma 7. *Let λ, μ and ν be partitions. Then $c^{\lambda}_{\nu/\mu}(q) = \langle S_\lambda \cdot \mu, \nu\rangle$.*

Proof. We have by definition

$$\mathcal{G}_{\nu/\mu}(X;q) = \sum_{\rho} \mathcal{K}_{\nu/\mu,\rho}(q) m_\rho = \sum_{\lambda} \left(\sum_{\rho} \mathcal{K}_{\nu/\mu,\rho}(q)\kappa_{\lambda\rho} \right) s_\lambda.$$

Thus $c^{\lambda}_{\nu/\mu}(q) = \sum_\rho \mathcal{K}_{\nu/\mu,\rho}(q)\kappa_{\lambda\rho}$. By standard results in symmetric function theory we also have $s_\lambda = \sum_\rho \kappa_{\lambda\rho} h_\rho$. Hence using the definitions of $\mathcal{V}_k$,

$$S_\lambda|\mu\rangle = \sum_{\rho} \kappa_{\lambda\rho}\mathcal{V}_\rho|\mu\rangle = \sum_{\nu} \left(\sum_{\rho} \kappa_{\lambda\rho}\mathcal{K}_{\nu/\mu,\rho}(q) \right) |\nu\rangle = \sum_{\nu} c^{\lambda}_{\nu/\mu}(q)|\nu\rangle.$$

□

8. The map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$

Define a representation $\Theta^* : H \to \mathrm{End}(\Lambda(q))$ by

$$B_k \longmapsto \begin{cases} k\frac{\partial}{\partial p_k} & \text{for } k > 0 \\ \left(\frac{1-q^{2nk}}{1-q^{2k}}\right) p_k & \text{for } k < 0. \end{cases}$$

Definition 8. Let $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ be the linear over $\mathbb{C}(q)$ map defined by

$$|\lambda\rangle \longmapsto \mathcal{G}_{\lambda/\tilde{\lambda}}(X;q).$$

The map $\mathbf{\Phi}$ has remarkable properties. It appears to convert linear properties in $\mathbf{F}$ into algebraic properties in $\Lambda(q)$. In [LLT], the Fock space $\mathbf{F}$ was identified with Λ and $\mathbf{\Phi}$ called "the adjoint of the q-plethysm operator" though its properties were not studied there.

The following theorem is the central representation theoretic result of this paper and shows that $\mathbf{\Phi}$ can be thought of as a projection of $\mathbf{F}$ onto the Fock space $\mathbb{C}(q)[H_-]$ of H.

Theorem 9. *The map $\mathbf{\Phi}$ is a map of H-modules. More precisely,*

$$\mathbf{\Phi} \circ \Theta = \Theta^* \circ \mathbf{\Phi}$$

as maps from H to $\mathrm{Hom}_{\mathbb{C}(q)}(\mathbf{F}, \Lambda(q))$.

Proof. We will check that $\mathbf{\Phi} \circ \Theta(B_k) = \Theta^*(B_k) \circ \mathbf{\Phi}$ for each k. Abusing notation, we do not distinguish B_k and $\Theta(B_k)$ from now on.

Let δ be an n-core which we fix throughout and suppose $k \geq 1$. We will calculate the expression $B_k S_\lambda|\delta\rangle$ in two ways. By Lemma 7 we can write

$$B_k S_\lambda|\delta\rangle = \sum_{\mu \in \mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q) B_k|\mu\rangle = \sum_{\nu \in \mathcal{P}_\delta} \left(\sum_{\mu \in \mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q) \mathcal{X}^k_{\mu/\nu}(q) \right) |\nu\rangle.$$

On the other hand, we can compute $B_k S_\lambda$ within H. We note that $B_k|\delta\rangle = 0$, which follows from the definition of $\tilde{\mathcal{V}}_k$ and the fact we cannot remove any ribbons from the shape δ. Thus by Lemma 4, we have

$$B_k S_\lambda|\delta\rangle = \left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_\mu \chi^k_{\lambda/\mu} S_\mu|\delta\rangle$$

where the coefficients $\chi^k_{\lambda/\mu}$ are given by $p_k^\perp s_\lambda = \sum_\mu \chi^k_{\lambda/\mu} s_\mu$ in Λ. By Lemma 7 again, we find that this is equal to

$$\left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_\mu \chi^k_{\lambda/\mu} \sum_{\nu \in \mathcal{P}_\delta} c^\mu_{\nu/\delta}(q)|\nu\rangle = \sum_{\nu \in \mathcal{P}_\delta} \left(\left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_\mu \chi^k_{\lambda/\mu} c^\mu_{\nu/\delta}(q) \right) |\nu\rangle.$$

Equating coefficients of $|\nu\rangle$ we obtain

$$\left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_\mu \chi^k_{\lambda/\mu} c^\mu_{\nu/\delta}(q) = \sum_{\mu \in \mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q) \mathcal{X}^k_{\mu/\nu}(q). \tag{3}$$

We now calculate

$$\begin{aligned}
\left(\frac{1-q^{2nk}}{1-q^{2k}} \right) p_k \mathcal{G}_{\nu/\delta}(X;q) &= \left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_{\mu \in \mathcal{P}_\delta} c^\mu_{\nu/\delta}(q) p_k s_\mu \\
&= \left(\frac{1-q^{2nk}}{1-q^{2k}} \right) \sum_\mu c^\mu_{\nu/\delta}(q) \left(\sum_\lambda \chi^k_{\lambda/\mu} s_\lambda \right) \\
&= \sum_\lambda \left(\sum_{\mu \in \mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q) \mathcal{X}^k_{\mu/\nu}(q) \right) s_\lambda \quad \text{using Equation (3)} \\
&= \sum_{\mu \in \mathcal{P}_\delta} \mathcal{X}^k_{\mu/\nu}(q) \mathcal{G}_{\mu/\delta}(X;q),
\end{aligned}$$

which is equivalent to $\Theta^*(B_{-k}) \cdot \mathbf{\Phi}(|\nu\rangle) = \mathbf{\Phi}(\Theta(B_{-k}) \cdot |\nu\rangle)$. This is true for all $|\nu\rangle$ and proves the claim for $k < 0$. The other case follows similarly. □

We have proved Theorem 9 by a calculation expressing ribbon functions in the Schur basis. A similar calculation using other bases is certainly possible. Theorem 9 gives the following Corollary.

Corollary 10. *Let λ be a partition and δ a fixed n-core. In $\Lambda(q)$ we have*

$$s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X] = \sum_{\mu \in \mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q) \mathcal{G}_{\mu/\delta}(X;q) = \sum_{\mu \in \mathcal{P}_\delta,\ \nu \in \mathcal{P}} c^\lambda_{\mu/\delta}(q) c^\nu_{\mu/\delta}(q) s_\nu(X).$$

Proof. These are immediate consequences of Theorem 9 and Lemma 7 as $\mathbf{\Phi}(|\delta\rangle) = 1$ for an n-core δ. ☐

9. Ribbon Pieri formulae

Define the formal power series

$$H(t) = \prod_{i\geq 1}\prod_{k=0}^{n-1} \frac{1}{1-x_i q^{2k} t}\ ;\ E(t) = \prod_{i\geq 1}\prod_{k=0}^{n-1}\left(1 + x_i q^{2k} t\right).$$

These power series are completely natural in the context of Robinson-Schensted ribbon insertion where they are spin-weight generating functions for sets of ribbons ([ShW1, vL]). Suppressing the notation for n, we define symmetric functions $\mathbf{h}_k$ and $\mathbf{e}_k$ by

$$H(t) = \sum_k \mathbf{h}_k t^k\ ;\ E(t) = \sum_k \mathbf{e}_k t^k.$$

In plethystic notation, $\mathbf{h}_k = h_k[(1+q^2+\cdots+q^{2(n-1)})X]$ and $\mathbf{e}_k = e_k[(1+q^2+\cdots+q^{2(n-1)})X]$. The following theorem is an immediate consequence Theorem 9, the definitions of $\mathcal{V}_k$, $\tilde{\mathcal{V}}_k$ and the plethysm $h_k[(1+q^2+\cdots+q^{2(n-1)})X]$ and Theorem 3.

Theorem 11 (Ribbon Pieri Rule)**.** *Let λ be a partition with n-core δ. Then*

$$\mathbf{h}_k \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_{\mu/\delta}(X;q) \tag{4}$$

where the sum is over all partitions μ such that μ/λ is a horizontal n-ribbon strip with k ribbons. Also

$$\mathbf{e}_k \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{s(\mu/\lambda)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all partitions μ such that μ/λ is a vertical n-ribbon strip with k ribbons.

We can also obtain the two statements of Theorem 11 from each other via the involution ω_n of Section 14. Let $\mathrm{mspin}(\lambda)$ denote the maximum spin of a ribbon tableau of shape λ. By Theorem 11, we have

$$\mathbf{h}_k = \sum_\lambda q^{\mathrm{mspin}(\lambda)} \mathcal{G}_\lambda(X;q) \tag{5}$$

where the sum is over all λ with no n-core such that $|\lambda| = kn$ with no more than n rows.

Example 12. *Let $n = 3$, $k = 2$ and $\lambda = (3,1)$. Then*

$$\mathbf{h}_2\mathcal{G}_{(3,1)} = \mathcal{G}_{(9,1)} + q\mathcal{G}_{(6,2,2)} + q^2\mathcal{G}_{(4,4,2)} + q^2\mathcal{G}_{(6,1,1,1,1)} + q^3\mathcal{G}_{(3,3,2,1,1)} + q^4\mathcal{G}_{(3,2,2,2,1)}.$$

Setting $q = 1$ in $H(t)$ we see that $\mathbf{h}_k(X;1) = \sum_\alpha h_\alpha$ where the sum is over all compositions $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ satisfying $\alpha_0 + \cdots + \alpha_{n-1} = k$. We may thus interpret Theorem 11 at $q = 1$ in terms of the n-quotient as the following formula:

$$\left(\sum_\alpha h_\alpha\right) s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_\alpha \left(h_{\alpha_0} s_{\lambda^{(0)}}\right) \cdots \left(h_{\alpha_{n-1}} s_{\lambda^{(n-1)}}\right) \tag{6}$$

where the sum is over the same set of compositions as above. Note that the right hand side of (6) is indeed equal to the right hand side of (4) at $q = 1$ since a horizontal ribbon strip of size k is just a union of horizontal strips with total size k in the n-quotient.

Remark 2. Shimozono and White's semistandard version of Barbasch-Vogan domino insertion [ShW] can be used to prove Theorem 11 for $n = 2$. For $n > 2$, the easiest $k = 1$ case of Theorem 11 can also be shown using Shimozono-White *color to spin* ribbon insertion [ShW1]. Despite further progress on the combinatorics of ribbon insertion by van Leeuwen, a combinatorial proof of Theorem 11 is still missing.

It is clear that we also obtain lowering versions of the Pieri rules. If $h_k = f(p_1, p_2, \ldots)$ we know that the adjoint operator (with respect to the usual inner product) is $h_k^\perp = f(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots)$. Thus by Theorem 9 and the definition of $\mathcal{U}_k$ we have

Proposition 13 (Ribbon Pieri Rule – Lowering Version)**.** *Let λ be a partition with n-core δ and $k \geq 1$ be an integer. Then*

$$h_k^\perp \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{s(\lambda/\mu)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all μ such that λ/μ is a horizontal ribbon strip. Similarly,

$$e_k^\perp \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{s(\lambda/\mu)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all μ such that λ/μ is a vertical ribbon strip.

This is a spin version of a branching formula first observed by Schilling, Shimozono and White [SSW] (see Section 16).

10. Border ribbon strip tableaux

Definition 14. A *border ribbon strip* S is a connected skew shape λ/μ with a distinguished tiling by disjoint non-empty horizontal ribbon strips $S_1, \ldots, S_a$ such that the diagram $S_{+i} = \cup_{j \leq i} S_j$ is a valid skew shape for every i and for each connected component C of S_i we have

1. The set of ribbons $C \cup S_{i-1}$ do not form a horizontal ribbon strip. Thus C has to 'touch' S_{i-1} 'from below'.
2. No sub horizontal ribbon strip C' of C which can be added to S_{i-1} satisfies the above property. Since C is connected, this is equivalent to saying that only the rightmost ribbon of C touches S_{i-1}.

We further require that S_1 is connected. The *height* $h(S_i)$ of the horizontal ribbon strip S_i is the number of its components (two squares are connected if they share a side, but not if they only share a corner). The height $h(S)$ of the border ribbon strip is defined as $h(S) = (\sum_i h(S_i)) - 1$. The *size* of the border ribbon strip S is then the total number of ribbons in $\cup_i S_i$. A *border ribbon strip tableau* is a chain $T = \lambda_0 \subset \lambda_1 \cdots \subset \lambda_r$ of shapes together with a structure of a border ribbon strip for each skew shape λ_i/λ_{i-1}. The *type* of $T = \{\lambda_i\}$ is then the composition α with α_i equal to the size of λ_i/λ_{i-1}. The height $h(T)$ is the sum of the heights of the composite border ribbon strips. Define $\widetilde{\mathcal{X}}^\nu_{\mu/\lambda}(q) \in \mathbb{Z}[q]$ by

$$\widetilde{\mathcal{X}}^\nu_{\mu/\lambda}(q) = \sum_T (-1)^{h(T)} q^{s(T)}$$

summed over all border ribbon strip tableaux of shape μ/λ and type ν. We will show in Section 11 that $\widetilde{\mathcal{X}}^\nu_{\mu/\lambda}(q) = \mathcal{X}^\nu_{\mu/\lambda}(q)$.

Note that this definition reduces to the usual definition of a border strip and border strip tableau when $n = 1$, in which case all the horizontal strips T_i of a border ribbon strip must be connected.

Example 15. *Let $n = 2$ and $\lambda = (4,2,2,1)$. Suppose S is a border ribbon strip such that S_1 has shape $(7,5,2,1)/(4,2,2,1)$, and thus it has size 3 and spin 1. We will now determine all the possible horizontal ribbon strips which may form S_2. It suffices to find the possible connected components that may be added. The domino $(9,5,2,1)/(7,5,2,1)$ may not be added since its union with S_1 is a horizontal ribbon strip, violating the conditions of the definition. The domino strip $(8,8,2,1)/(7,5,2,1)$ is not allowed since the domino $(8,8,2,1)/(7,7,2,1)$ can be removed and we still obtain a strip which touches S_1.*

The connected horizontal ribbon strips C which can be added are $(7,7,2,1)/(7,5,2,1)$, $(7,5,3,3,2,1)/(7,5,2,1)$ and $(7,5,4,1)/(7,5,2,1)$ as shown in Figure 4. Thus assuming S_2 is non-empty, there are 5 choices for S_2, corresponding to taking some compatible combination of the three connected horizontal ribbon strips above.

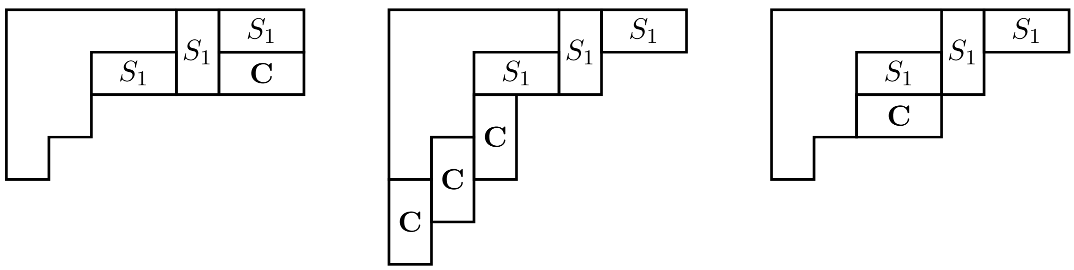

FIGURE 4. Connected horizontal strips C which can be added to $S_1 = (7,5,2,1)/(4,2,2,1)$ to form a border ribbon strip. The resulting border ribbon strips all have height 1.

Example 16. *Let $n = 2$. We will calculate $\widetilde{\mathcal{X}}^5_{\lambda/\mu}(q)$ for $\lambda = (5,5,2)$ and $\mu = (2)$. The relevant border ribbon strips S are (successive differences of the following chains denote the S_i)*

- $(2) \subset (5,5,2)$ *with height 0 and spin 5,*
- $(2) \subset (5,3,2) \subset (5,5,2)$ *with height 1 and spin 3,*
- $(2) \subset (5,5) \subset (5,5,2)$ *with height 1 and spin 3,*
- $(2) \subset (5,3) \subset (5,5,2)$ *with height 2 and spin 1.*

Thus $\widetilde{\mathcal{X}}^5_{\lambda/\mu}(q) = q^5 - 2q^3 + q$.

The condition on a horizontal ribbon strip to be connected can be described in terms of the n-quotient as follows. Let T be a ribbon tableau with n-quotient $\{T^{(0)}, \ldots, T^{(n-1)}\}$. Let $\{(d_i, p_i)\}$ be the set of diagonals which are nonempty in the n-quotient of the horizontal ribbon strip R: thus diagonal diag_{d_i} of $T^{(p_i)}$ contains a square corresponding to some ribbon in the horizontal ribbon strip R. Then the horizontal ribbon strip R is connected if and only if the set of integers $\{d_i\}$ is an interval (connected) in $\mathbb{Z}$. Thus border ribbon strips may be characterised in terms of the n-quotient.

11. Formal relationship between Murnaghan-Nakayama and Pieri rules

Let V be a vector space over $\mathbb{C}(q)$ and $\{v_\lambda\}_{\lambda \in \mathcal{P}}$ be a set of vectors in V labelled by partitions. Suppose $\{P_k\}$ are commuting linear operators satisfying

$$P_k v_\lambda = \sum_\mu \widetilde{\mathcal{X}}^k_{\mu/\lambda}(q) v_\mu \quad \text{for all } k, \tag{7}$$

then we will say that the ribbon Murnaghan-Nakayama rule holds for $\{P_k\}$. Suppose $\{H_k\}$ are commuting linear operators on V satisfying

$$H_k v_\lambda = \sum_{\mu} \mathcal{K}_{\mu/\lambda,k}(q) v_\mu \quad \text{for all } k, \tag{8}$$

then we will say that the ribbon Pieri formula holds for $\{H_k\}$.

If the skew shapes μ/λ are replaced by λ/μ in the above formulae, we get adjoint versions of these formulae which can be thought of as lowering operator formulae. Thus if a set of commuting linear operators $\{P_k^*\}$ satisfies

$$P_k^* v_\lambda = \sum_{\mu} \widetilde{\mathcal{X}}^k_{\lambda/\mu}(q) v_\mu \quad \text{for all } k,$$

then we will say the lowering ribbon Murnaghan-Nakayama rule holds, and similarly for the lowering ribbon Pieri rule. We begin by observing the following easy lemma.

Lemma 17. *The power sum and homogeneous symmetric functions satisfy:*

$$mh_m = p_{m-1}h_1 + p_{m-2}h_2 + \cdots + p_m.$$

Proof. See (2.10) in [Mac]. □

Theorem 18. *Fix $n \geq 1$ as usual. Let $\{H_k\}$ and $\{P_k\}$ be commuting sets of linear operators, acting on a $\mathbb{C}(q)$-vector space V, satisfying the relations of Lemma 17 between h_k and p_k in Λ. Then the ribbon Murnaghan-Nakayama rule (7) holds for $\{P_k\}$ if and only if the ribbon Pieri rule (8) holds for $\{H_k\}$ (with respect to the same set of vectors $\{v_\lambda\}_{\lambda\in\mathcal{P}}$). An analogous statement holds for the lowering versions of the respective rules.*

Proof. Let us suppose that (7) holds. We will proceed by induction on k. Since $H_1 = P_1$ and a border ribbon strip of size 1 is exactly the same as a horizontal ribbon strip of size 1, the starting condition is clear. Now suppose the proposition has been shown up to $k-1$. By assumption, kH_k acts on V in the same way that $H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k$ does.

We first consider the coefficient of v_μ in $(H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k) \cdot v_\lambda$ by formally applying the rules (7) and (8). We obtain one term for each pair (S, T) where S is a border ribbon strip of size between 1 and k satisfying $sh(S) = \nu/\lambda$ (for some ν) and T is a horizontal ribbon strip of size $k - \text{size}(S)$ satisfying $sh(T) = \mu/\nu$. Denote by $(S_1, \ldots, S_a)$ the distinguished decomposition of S into horizontal ribbon strips.

Construct a directed graph $G_{\lambda,\mu,k}$ with vertices labelled by such pairs $\mathcal{S} = \{(S,T)\}$. We have an edge

$$(S,T) \longrightarrow (S - S_a, T \cup S_a) \tag{9}$$

for every pair (S,T) such that $a > 1$ and $T \cup S_a$ is a horizontal strip (with the induced tiling). We claim that every non isolated connected component W of $G_{\lambda,\mu,k}$ is an inward pointing star. Indeed, every vertex must have outdegree or indegree equal to 0, and the maximum outdegree is 1, since by Condition 1 of Definition 14 the right hand vertex of (9) has outdegree 0.

Let us consider a vertex (S', T') (where $S' = \{S'_1, \ldots, S'_a\}$) with non-zero indegree. Now let C be a component of T' such that $C \cup S'_a$ is not a horizontal ribbon strip. Then there is a unique sub-horizontal ribbon strip C' of C which can be added to S' to form a border ribbon strip, by Condition 2 of Definition 14. This C' may be described as follows. Order the ribbons of C from left to right $c_1, c_2, \ldots, c_l$. Find the smallest i such that c_i touches the bottom of S'_a and we set $C' = \{c_1, c_2, \ldots, c_i\}$. We call such a horizontal ribbon strip C' an *addable strip* of T' (with respect to S').

A non-isolated connected component $W_{(S',T')}$ of $G_{\lambda,\mu,k}$ contains exactly of such a vertex (S',T') together with the pairs (S,T) such that $S=\{S'_1,\ldots,S'_a,S_{a+1}\}$, and S_{a+1} is the union of some (arbitrary) subset of the set of addable strips of T'. It is immediate from the construction that (S,T) will be a valid pair in $\mathcal{S}$. The contribution of $W_{(S',T')}$ to the coefficient of v_μ in $(H_{k-1}P_1+H_{k-2}P_2+\cdots+P_k)\cdot v_\lambda$ is

$$\sum_{(S,T)\in W_{(S',T')}}(-1)^{h(S)}q^{s(S\cup T)}=(-1)^{h(S')}q^{s(S'\cup T')}\sum_{\{C'\}}(-1)^{|\{C'\}|}$$

where on the right hand side, $\{C'\}$ varies over arbitrary subsets of addable strips of T' (we have used the fact that the tiling never changes so the spin is constant, together with the definition of height). This contribution is 0, corresponding to the identity $(1-1)^c=0$ where c is the number of addable strips of T'.

It remains to consider the contribution of the isolated vertices: these are pairs (S,T) where $S=(S_1)$ is a connected horizontal ribbon strip such that $S\cup T$ is also a horizontal ribbon strip. Since S is connected we can recover it from $S\cup T$ by specifying its rightmost ribbon, by Condition 1 of Definition 14. Thus such pairs occur exactly k times for each horizontal ribbon strip of shape μ/λ, and hence the ribbon Pieri rule (8) is satisfied for the operator H_k.

The converse and dual claims follow from the same argument. □

12. The ribbon Murnaghan-Nakayama rule

It is now clear that the action of the bosonic operators B_k on $\mathbf{F}$ can be described in terms of border ribbon strips.

Theorem 19. *We have* $\mathcal{X}^k_{\lambda/\mu}(q)=\widetilde{\mathcal{X}}^k_{\lambda/\mu}(q)$.

Proof. The operators $\mathcal{V}_k$ commute and satisfy the ribbon Pieri rule (8) with respect to the basis $\{|\lambda\rangle\}$, by definition. The claim follows from Theorem 18 applied to $V=\mathbf{F}$ and $v_\lambda=|\lambda\rangle$. □

The next theorem is a ribbon analogue of the classical Murnaghan-Nakayama rule which calculates the characters of the symmetric group.

Theorem 20 (Ribbon Murnaghan-Nakayama Rule)**.** *Let $k\geq 1$ be an integer and ν be a partition with n-core δ. Then*

$$\left(1+q^{2k}+\cdots+q^{2k(n-1)}\right)p_k\mathcal{G}_{\nu/\delta}(X;q)=\sum_\mu\widetilde{\mathcal{X}}^k_{\mu/\nu}(q)\mathcal{G}_{\mu/\delta}(X;q). \tag{10}$$

Also

$$k\frac{\partial}{\partial p_k}\mathcal{G}_{\nu/\delta}(X;q)=\sum_\mu\widetilde{\mathcal{X}}^k_{\nu/\mu}(q)\mathcal{G}_{\mu/\delta}(X;q).$$

Proof. The theorem follows from Theorems 18 and 11, where $V=\Lambda(q)$ and $v_\lambda=\mathcal{G}_{\lambda/\delta}(X;q)$. □

It is rather difficult to interpret Theorem 20 in terms of the n-quotient at $q=1$. When $q=1$ the product $\left(1+q^{2k}+\cdots+q^{2k(n-1)}\right)p_k\mathcal{G}_{\lambda/\delta}(X;q)$ becomes $np_ks_{\lambda^{(0)}}s_{\lambda^{(1)}}\cdots s_{\lambda^{(n-1)}}$ which may be written as the sum of n usual Murnaghan-Nakayama rules as

$$\sum_{i=0}^{n-1}s_{\lambda^{(0)}}\cdots\left(p_ks_{\lambda^{(i)}}\right)\cdots s_{\lambda^{(n-1)}}.$$

Thus we might expect that border ribbon strips of size k correspond to adding a usual ribbon strip of size k to one partition in the n-quotient. However, the following example shows that this cannot work.

Example 21. *By the ribbon Murnaghan-Nakyama rule (Theorem 20) with $k = n = 2$ and $\nu = \emptyset$,*

$$(1 + q^4)p_2 \cdot 1 = \mathcal{G}_{(4)} + q\mathcal{G}_{(3,1)} + (q^2 - 1)\mathcal{G}_{(2,2)} - q\mathcal{G}_{(2,1,1)} - q^2\mathcal{G}_{(1,1,1,1)}.$$

We can compute directly that

$$\mathcal{G}_{(4)} = h_2, \quad \mathcal{G}_{(3,1)} = qh_2, \quad \mathcal{G}_{(2,1,1)} = qe_2$$
$$\mathcal{G}_{(2,2)} = q^2h_2 + e_2, \quad \mathcal{G}_{(1,1,1,1)} = q^2e_2,$$

verifying Theorem 20 directly. On the other hand, the shapes which correspond to a single border strip in one partition of the 2-quotient are $\{(4), (3,1), (2,1,1), (1,1,1,1)\}$ and the corresponding $\mathcal{G}_\lambda$ terms do not give $(1 + q^4)p_2$.

It seems possible that the ribbon Murnaghan-Nakayama rule may have some relationship with the representation theory of the wreath products $S_n \mathrm{wr} C_p$, or even more likely to the cyclotomic Hecke algebras associated to these wreath products (see for example [Mat]).

13. An involution on F

Following [LT], define a semi-linear involution $v \mapsto v'$ on $\mathbf{F}$ by $q' = q^{-1}$ and

$$|\lambda\rangle \longmapsto |\lambda'\rangle.$$

Then we have [LT, Proposition 7.10]

Proposition 22. *For all $u \in \mathbf{F}$ and compositions β satisfying $|\beta| = k$ we have*

$$(\mathcal{V}_\beta u)' = q^{-(n-1)k}\tilde{\mathcal{V}}_\beta u', \qquad (\mathcal{U}_\beta u)' = q^{-(n-1)k}\tilde{\mathcal{U}}_\beta u'.$$

Proof. We use the descriptions of the action of $\mathcal{V}_k$ and $\tilde{\mathcal{V}}_k$ in terms of horizontal and vertical ribbon strips, together with the calculation $s(T) + s(T') = (n-1) \cdot r$ for a ribbon tableau T and its conjugate T' which contain r ribbons. □

14. The ribbon involution ω_n

In this section we will define an involution w_n on $\Lambda(q)$ which is essentially the image of the involution $v \mapsto v'$ on the Fock space $\mathbf{F}$ of Section 13. However, this involution will turn out to be not just a semi-linear involution, but also a $\mathbb{C}$-algebra isomorphism of $\Lambda(q)$.

Definition 23. Define the *ribbon involution* $w_n : \Lambda(q) \to \Lambda(q)$ as the semi-linear map satisfying $w_n(q) = q^{-1}$ and

$$w_n(s_\lambda) = q^{(n-1)|\lambda|}s_{\lambda'}.$$

Theorem 24. *The map w_n is a $\mathbb{C}$-algebra homomorphism which is an involution. It maps $\mathcal{G}_{\lambda/\mu}$ into $\mathcal{G}_{(\lambda/\mu)'}$ for every skew shape λ/μ.*

Proof. The fact that w_n is an algebra homomorphism follows from the fact that if $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$ then $s_{\lambda'} s_{\mu'} = \sum c^\nu_{\lambda\mu} s_{\nu'}$, and that the grading is preserved by multiplication. That w_n is an involution is a quick calculation.

For the last statement, we use Proposition 22 and the fact that the involution $w(h_n) = e_n$ satisfies $w(s_\lambda) = s_{\lambda'}$ to obtain $(S_\nu|\mu\rangle)' = q^{-(n-1)k} S_{\nu'}|\mu'\rangle$. By Lemma 7 this implies that

$$\sum_\lambda c^\nu_{\lambda/\mu}(q^{-1})|\lambda'\rangle = q^{-(n-1)k} \sum_\lambda c^{\nu'}_{\lambda'/\mu'}(q)|\lambda'\rangle.$$

Here $k = |\nu|$. Equating coefficients of $|\lambda'\rangle$ we obtain $c^\nu_{\lambda/\mu}(q^{-1}) = q^{-(n-1)k} c^{\nu'}_{\lambda'/\mu'}(q)$. Thus

$$w_n(\mathcal{G}_{\lambda/\mu}) = \sum_\nu w_n(c^\nu_{\lambda/\mu}(q)s_\nu) = \sum_\nu \left(c^{\nu'}_{\lambda'/\mu'}(q) q^{-(n-1)|\nu|}\right) q^{(n-1)|\nu|} s_{\nu'} = \mathcal{G}_{\lambda'/\mu'}.$$

□

Proposition 25. *Let $f \in \Lambda(q)$ have degree k. Then we have*

$$q^{2(n-1)k}\omega_n\left(\Upsilon_{q,n}(f)\right) = \Upsilon_{q,n}\left(\omega_n(f)\right).$$

In particular, if $\lambda \vdash k$ we have

$$\omega_n\left(s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]\right) = q^{-(n-1)k} s_{\lambda'}[(1+q^2+\cdots+q^{2(n-1)})X].$$

Proof. Since both ω_n and $\Upsilon_{q,n}(f)$ are $\mathbb{C}$-algebra homomorphisms we need only check this for the elements p_k and for q, for which the computation is straightforward. □

15. The ribbon Cauchy identity

Define the formal power series $\Omega_n(XY;q)$ and $\tilde{\Omega}_n(XY;q)$ by

$$\Omega_n(XY;q) = \prod_{i,j}\prod_{k=0}^{n-1} \frac{1}{1-x_iy_jq^{2k}} \quad ; \quad \tilde{\Omega}_n(XY;q) = \prod_{i,j}\prod_{k=0}^{n-1}\left(1+x_iy_jq^{2k}\right).$$

A combinatorial proof via ribbon insertion of the following identity was given by van Leeuwen [vL].

Theorem 26 (Ribbon Cauchy Identity)**.** *Fix n as usual and a n-core δ. Then*

$$\Omega_n(XY;q) = \sum \mathcal{G}_{\lambda/\delta}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q)$$

where the sum is over all λ such that $\tilde{\lambda} = \delta$.

Unlike for the Schur functions, this does not imply that the $\mathcal{G}_{\lambda/\delta}$ form an orthonormal basis under a certain inner product, as they are not linearly independent.

Proof. By Corollary 10 we have

$$s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X] = \sum_{\mu\in\mathcal{P}_\delta} c^\lambda_{\mu/\delta}(q)\mathcal{G}_{\mu/\delta}(X;q).$$

Thus

$$\begin{aligned}\sum_\lambda s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]s_\lambda(Y) &= \sum_{\mu\in\mathcal{P}_\delta}\left(\sum_\lambda c^\lambda_{\mu/\delta}(q)s_\lambda(Y)\right)\mathcal{G}_{\mu/\delta}(X;q)\\ &= \sum_{\mu\in\mathcal{P}_\delta}\mathcal{G}_{\mu/\delta}(X;q)\mathcal{G}_{\mu/\delta}(Y;q).\end{aligned}$$

Let $\Upsilon_{q,n}(X)$ denote the algebra automorphism of $\Lambda[X](q)\otimes_{\mathbb{C}(q)}\Lambda[Y](q)$ given by applying $\Upsilon_{q,n}$ to the X variables only. Applying $\Upsilon_{q,n}(X)$ to $\log\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right) = \sum_k \frac{1}{n}p_k(X)p_k(Y)$

gives $\log\left(\prod_{i,j}\prod_{k=1}^{n-1}\frac{1}{1-x_iy_jq^{2k}}\right)$ which is exactly $\log(\Omega_n)$. Thus applying $\Upsilon_{q,n}(X)$ to the usual Cauchy identity for Schur functions ($\prod_{i,j}\frac{1}{1-x_iy_j}=\sum_\lambda s_\lambda(X)s_\lambda(Y)$) gives

$$\Omega_n(XY;q)=\sum_\lambda s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]s_\lambda(Y)$$

from which the Theorem follows. □

Now let us compute $\omega_n(\Omega)$ where we let $\omega_n:\Lambda[X](q)\otimes_{\mathbb{C}(q)}\Lambda[Y](q)\to\Lambda[X](q)\otimes_{\mathbb{C}(q)}\Lambda[Y](q)$ act on the X variables by

$$\omega_n(f(X;q)\otimes g(Y;q))\longmapsto\omega_n(f(X;q))\otimes g(Y;q^{-1}).$$

One checks immediately that this is indeed an algebra involution. We have (fixing an n-core δ)

$$\omega_n(\Omega_n)=\sum_{\lambda\in\mathcal{P}_\delta}\mathcal{G}_{\lambda'/\delta'}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q^{-1}).$$

Also,

$$\omega_n(\Omega_n)=\sum_\lambda q^{(n-1)|\lambda|}s_{\lambda'}(X)s_\lambda[(1+q^{-2}+\cdots+q^{-2(n-1)})Y]=\prod_{i,j}\prod_{k=0}^{n-1}(1+x_iy_jq^{n-1-2k}).$$

Thus

$$\sum_{\lambda\in\mathcal{P}_\delta}\mathcal{G}_{\lambda'/\delta'}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q^{-1})=\prod_{i,j}\prod_{k=0}^{n-1}(1+x_iy_jq^{n-1-2k}).$$

If we multiply the d^{th} graded piece of each side by $q^{(n-1)d}$ we obtain the following result.

Proposition 27 (Dual Ribbon Cauchy Identity)**.** *Fix an n-core δ. We have*

$$\tilde{\Omega}_n(XY;q)=\sum_{\lambda\in\mathcal{P}_\delta}q^{(n-1)|\lambda/\tilde{\lambda}|}\mathcal{G}_{\lambda'/\delta'}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q^{-1}).$$

The factor of $q^{(n-1)|\lambda/\tilde{\lambda}|}$ can be explained combinatorially by the fact that the spins $s(T)$ and $s(T')$ of a ribbon tableau T and its conjugate T' satisfy $s(T')=q^{(n-1)|\lambda/\tilde{\lambda}|}-s(T)$.

16. Skew and super ribbon functions

We now describe some properties of the skew ribbon functions $\mathcal{G}_{\lambda/\mu}(X;q)$. Unfortunately, we have been unable to describe them in analogy with the formula $s_{\lambda/\mu}=s_\lambda^\perp s_\mu$. However, we do have the following *skew ribbon Cauchy identity.*

Proposition 28. *Let μ be any partition. Then*

$$\mathcal{G}_{\mu/\tilde{\mu}}(X;q)\Omega_n(XY;q)=\sum_\lambda\mathcal{G}_{\lambda/\tilde{\lambda}}(X;q)\mathcal{G}_{\lambda/\mu}(Y;q)$$

where the sum is over all λ satisfying $\tilde{\lambda}=\tilde{\mu}$.

Proof. Lemma 7 and Theorem 9 imply that

$$s_\nu[(1+q^2+\ldots+q^{2(n-1)})X]\mathcal{G}_{\mu/\tilde{\mu}}(X;q)=\sum_\lambda c^\nu_{\lambda/\mu}(q)\mathcal{G}_{\lambda/\tilde{\lambda}}(X;q).$$

Now multiply both sides by $s_\nu(Y)$ and sum over ν. Finally use Theorem 26. □

Schilling, Shimozono and White [SSW] have also used skew ribbon functions, as follows (their original result used cospin rather than spin). By the combinatorial definition of $\mathcal{G}_\lambda$ we immediately have the coproduct expansion $\mathcal{G}_\lambda(X+Y;q) = \sum_\mu \mathcal{G}_\mu(X;q)\mathcal{G}_{\lambda/\mu}(Y;q)$. Since (see [Mac]), $\Delta f = \sum_\mu s_\mu^\perp f \otimes s_\mu$ we get immediately that

$$s_\nu^\perp \mathcal{G}_\lambda(X;q) = \sum_\mu \mathcal{G}_\mu(X;q) \left\langle \mathcal{G}_{\lambda/\mu}(Y;q), s_\nu \right\rangle.$$

Setting $\nu = (k)$ we obtain the lowering version of the Pieri rule (Proposition 13).

Another related generalisation of the usual ribbon functions are super ribbon functions. Fix a total order '$\prec$' on two alphabets $A = \{1 < 2 < 3 < \cdots\}$ and $A' = \{1' < 2' < 3' < \cdots\}$ (which we assume to be compatible with each of their natural orders). For example, one could pick $1 \prec 1' \prec 2 \prec 2' \prec \cdots$.

Definition 29. A *super ribbon tableau* T of shape λ/μ is a ribbon tableau of the same shape with ribbons labelled by the two alphabets such that the ribbons labelled by a for $a \in A$ form a horizontal ribbon strip and those labelled by a' for $a' \in A'$ form a vertical ribbon strip. These strips are required to be compatible with the chosen total order. Thus the shape obtained by removing ribbons labelled by elements $\succ i$ must be a skew shape $\lambda_{\preceq i}/\mu$, for each $i \in A \cup A'$.

Define the *super ribbon function* $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ as the following generating function:

$$\mathcal{G}_{\lambda/\mu}(X/Y;q) = \sum_T q^{s(T)} x^{w(T)} (-y)^{w'(T)}$$

where the sum is over all super ribbon tableaux T of shape λ/μ and $w(T)$ is the weight in the first alphabet A while $w'(T)$ is the weight in the second alphabet A'. For a composition α, we use $(-y)^\alpha$ to stand for $(-y_1)^{\alpha_1}(-y_2)^{\alpha_2}\cdots(-y_l)^{\alpha_l}$.

Proposition 30. *The super ribbon function $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ is a symmetric function in the X and Y variables, separately. It does not depend on the total order on the alphabets A and A'.*

Proof. If we pick the total order on $A \cup A'$ to be so that $a > a'$ for any $a \in A$ and $a' \in A'$ then we have $[x^\alpha(-y)^\beta]\mathcal{G}_{\lambda/\mu}(X/Y;q) = \left\langle \mathcal{V}_\alpha \tilde{\mathcal{V}}_\beta \mu, \lambda \right\rangle$ for any compositions α and β. The proof of symmetry is completely analogous to that of Theorem 6, using the commutativity of both the operators $\{\mathcal{V}_k\}$ and $\{\tilde{\mathcal{V}}_k\}$. The last statement requires the fact that $\{\mathcal{V}_k\}$ commutes with $\{\tilde{\mathcal{V}}_k\}$. ∎

17. The ribbon inner product and the bar involution on $\Lambda(q)$

Definition 31. Let $\langle .,. \rangle_n : \Lambda(q) \times \Lambda(q) \to \mathbb{C}(q)$ be the $\mathbb{C}(q)$-bilinear map defined by

$$\left\langle p_\lambda[(1 + q^2 + \cdots + q^{2(n-1)})X], p_\mu \right\rangle = z_\lambda \delta_{\lambda\mu}.$$

It is clear that $\langle .,. \rangle_n$ is non-degenerate. The inner product $\langle .,. \rangle_n$ is related to Ω_n in the same way as the usual inner product is related to the usual Cauchy kernel – the following claim is immediate.

Proposition 32. *Two bases $\{v_\lambda\}$ and $\{w_\lambda\}$ of $\Lambda(q)$ are dual with respect to $\langle .,. \rangle_n$ if and only if*

$$\sum_\lambda v_\lambda(X) w_\lambda(Y) = \Omega_n.$$

In particular, $\left\{s_\lambda[(1+q^2+\ldots+q^{2(n-1)})X]\right\}$ *is dual to* $\{s_\lambda\}$.

Lemma 33. *The inner product* $\langle .,.\rangle_n$ *is symmetric.*

Proof. This is clear from the definition as we can just check this on the basis p_λ of $\Lambda(q)$. □

Recall that for $f \in \Lambda$, $f^\perp$ denotes its adjoint with respect to the Hall inner product.

Lemma 34. *The operator* $f^\perp$ *is adjoint to multiplication by* $\Upsilon_{q,n}(f) \in \Lambda(q)$.

Proof. This is a consequence of $\langle f, g\rangle = \langle \Upsilon_{q,n}(f), g\rangle_n$. □

The inner product $\langle .,.\rangle_n$ is compatible with the inner product $\langle |\lambda\rangle, |\mu\rangle\rangle = \delta_{\lambda\mu}$ on $\mathbf{F}$ when we restrict our attention to the space of highest weight vectors of $U_q(\widehat{\mathfrak{sl}}_n)$.

Proposition 35. *Let* $u, v \in \mathbf{F}$ *be highest weight vectors for the action of* $U_q(\widehat{\mathfrak{sl}}_n)$. *Then* $\langle \mathbf{\Phi}(u), \mathbf{\Phi}(v)\rangle_n = \langle u, v\rangle$.

Proof. We check the claim for the basis $\{B_{-\lambda}|0\rangle\}$ of the space of highest weight vectors in $\mathbf{F}$. □

Our next definition is the image of the bar involution of the Fock space $^-: \mathbf{F} \to \mathbf{F}$ ([LT]). Using this involution, Lascoux, Leclerc and Thibon [LLT1, LT, LT1] have studied global bases of $\mathbf{F}$.

Definition 36. Define the $\mathbb{C}$-algebra involution $^-: \Lambda(q) \to \Lambda(q)$ by $\overline{q} = q^{-1}$ and

$$p_k \longmapsto q^{2(n-1)k} p_k.$$

It is clear that $^-$ is indeed an involution. We have the following basic properties of $^-$, imitating similar properties in $\mathbf{F}$ ([LT, Theorem 7.11]).

Proposition 37. *Let* $u, v \in \Lambda(q)$. *The involution* $^-: \Lambda(q) \to \Lambda(q)$ *has the following properties:*

$$\overline{\Upsilon_{q,n}(p_k)} = \Upsilon_{q,n}(p_k),$$

$$\langle \overline{u}, v\rangle_n = \left\langle \omega_n(u), \overline{\omega_n(v)} \right\rangle_n.$$

Proof. The first statement is a straightforward computation. For the second statement, we compute explicitly both sides for the basis p_λ of $\Lambda(q)$. □

Proposition 37 and results in [LT] show that $\overline{\mathbf{\Phi}(v)} = \mathbf{\Phi}(\overline{v})$ for all u, v in the subspace of highest weight vectors in $\mathbf{F}$. However this is not true in general. For example, $|(3,1)\rangle + q|(2,2)\rangle + q^2|(2,1,1)\rangle$ is bar invariant in $\mathbf{F}$ but its image under $\mathbf{\Phi}$ is not.

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