ON SYMMETRY AND POSITIVITY FOR DOMINO AND RIBBON TABLEAUX

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Abstract. Inspired by the spin-inversion statistic of Schilling, Shimozono and White [9] and Haglund et al. [2] we relate the symmetry of ribbon functions to a result of van Leeuwen, and also describe the multiplication of a domino function by a Schur function.

1. Introduction

Lascoux, Leclerc and Thibon [6] defined spin-weight generating functions $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ (from hereon called *ribbon functions*) for ribbon tableaux. They showed that these functions were symmetric functions with coefficients in $\mathbb{C}(q)$ using the action of the Heisenberg algebra on the Fock space of $U_q(\widehat{\mathfrak{sl}}_n)$. Later, Leclerc and Thibon [5] showed the expansion coefficients of $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ in terms of Schur functions were parabolic Kazhdan-Lusztig polynomials for the affine Hecke algebra of type A.

For the $n = 2$ case of domino tableaux, a combinatorial proof of the symmetry and in fact a description of the expansion of $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ in terms of Schur functions is given by the Yamanouchi domino tableaux of Carré and Leclerc [1]. More recently, Schilling, Shimozono and White [9] and separately Haglund et. al. [2] have described the spin statistic of a ribbon tableau in terms of an inversion number on the n-quotient. This article gives two applications of this inversion number towards the ribbon functions.

Our first application is a proof of the symmetry of ribbon functions using a result of van Leeuwen [7] developed from his spin-preserving Knuth correspondence for ribbon tableaux. The result says roughly that the spin generating functions for adding horizontal ribbon strips above or below a lattice path vertical on both ends are equal. Another "elementary" but more systematic proof of the symmetry of ribbon functions will appear in [3]. We also describe explicitly a bijection in terms of words required to prove the symmetry of ribbon functions.

Our second application is an imitation of Stembridge's concise proof of the Littlewood Richardson rule [10] for the domino tableau case. We describe the expansion of $s_\nu(X)\mathcal{G}^{(2)}_{\mu/\rho}(X;q)$ in the basis of Schur functions in terms of ν-*Yamanouchi domino tableaux*. This description appears to be new and also gives a shorter proof of the result of Carré and Leclerc [1], corresponding to $\nu = (0)$, the empty partition.

We refer the reader to [6, 4] for the necessary definitions and notation concerning ribbon tableaux, spin and ribbon functions. We will always think of our partitions and tableaux as being drawn in the English notation.

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2. Spin-inversion statistic

We will use the spin-inversion statistic from [2] as its description is considerably shorter than the one in [9], and we will only be interested in how spin changes rather than its exact value. Let $\mathrm{quot}_n(T) = (T^{(0)}, \ldots, T^{(n-1)})$ denote the n-quotient of a ribbon tableau T (which may have skew shape). With the n-core fixed, semistandard ribbon tableaux are in bijection with such n-tuples of usual semistandard Young tableaux. The diagonal $\mathrm{diag}(s)$ of a cell $s \in \mathrm{quot}_n(T)$ is equal to the diagonal of T on which the head of the corresponding ribbon $\mathrm{Rib}(s)$ lies. For a cell $s \in T^{(i)}$ it is given by $\mathrm{diag}(s) = nc(s) + c_i$ for some offsets c_i depending on the n-core of $\mathrm{sh}(T)$. Here $c(s) = j - i$ is the usual *content* of a square $s = (i, j)$. An *inversion* is a pair of entries $T(x) = a, T(y) = b$ such that $a < b$ and $0 < \mathrm{diag}(x) - \mathrm{diag}(y) < n$. We denote by $\mathrm{inv}(T) = \mathrm{inv}(\mathrm{quot}_n(T))$ the number of inversions of $\mathrm{quot}_n(T)$. We have [2]

Lemma 1. *Given a skew shape λ/μ, there is a constant $e(\lambda/\mu)$ such that for every standard n-ribbon tableau T of shape λ/μ, we have* $\mathrm{spin}(T) = e(\lambda/\mu) - \mathrm{inv}(\mathrm{quot}_n(T))$.

We shall use a particular *diagonal reading order* on our tableaux. Let T be a ribbon tableau. The *reading word* $\mathrm{r}(T)$ is given by reading the diagonals of $\mathrm{quot}_n(T)$ in descending order, where in each diagonal the larger numbers are read first. We will regularly abuse notation by allowing ourselves to identify ribbons in T with squares of the n-quotient $\mathrm{quot}_n(T)$. We will also identify a skew shape λ/μ which is a horizontal ribbon strip with the correpsonding horizontal ribbon strip tableau T satisfying $\mathrm{sh}(T) = \lambda/\mu$.

3. Symmetry of ribbon functions

We fix the length $n \geq 1$ of our ribbons throughout.

Recall that the standard way to prove that a Schur function is symmetric is to give involutions α_i on semistandard tableaux of shape λ which swaps the number of i's and $(i+1)$'s, for each i. This is known as the Bender-Knuth involution. Our first aim is to study the symmetry of the ribbon functions $\mathcal{G}^{(n)}_{\lambda/\mu}(X; q)$ from the perspective of the n-quotient. A *ribbon Bender-Knuth involution* σ_i is any shape and spin preserving involution on ribbon tableaux T which changes the number of i's and $i+1$'s while keeping all other values unchanged. One can define an involution σ'_i on ribbon tableaux $T = (T^{(0)}, \ldots, T^{(n-1)})$ by $\sigma'_i(T) = \sigma'_i(T^{(0)}, \ldots, T^{(n-1)}) = (\alpha_i(T^{(0)}), \ldots, \alpha_i(T^{(n-1)}))$. Unfortunately, this involution σ'_i is not spin-preserving and a ribbon Bender-Knuth involution is necessarily more complicated.

We call a skew shape λ/μ a double horizontal ribbon strip if it can be tiled by two horizontal ribbon strips. Let $\mathcal{R}^{a,b}_{\lambda/\mu}$ be the set of ribbon tableaux of shape λ/μ filled with a 1's and b 2's. To obtain a ribbon Bender-Knuth involution, it suffices to find a spin preserving bijection between $\mathcal{R}^{a,b}_{\lambda/\mu}$ and $\mathcal{R}^{b,a}_{\lambda/\mu}$ for every a and b and every double horizontal strip λ/μ. Let $T \in \mathcal{R}^{a,b}_{\lambda/\mu}$. Suppose some tableau $T^{(i)}$ of the n-quotient contains a column with two squares, then those two squares must be 1 on top of a 2.

We first show that we may reduce to the case that the n-quotient contains no such columns. If (x, y) is an inversion of T we say that the inversion *involves* x and y. Let $\mathrm{inv}_x(T)$ denote the number of inversions of T which involve x.

Lemma 2. *Let T be a ribbon tableau and $\mathrm{quot}_n(T)$ contain two squares x and y in the same column such that $T(x) = i$ and $T(y) = i+1$. Let T' be a semistandard ribbon tableau obtained from T by changing an "i" to an "$i+1$". Then*

$$\mathrm{inv}_x(T) + \mathrm{inv}_y(T) = \mathrm{inv}_x(T') + \mathrm{inv}_y(T').$$

Proof. We first note that $\mathrm{diag}(x) = \mathrm{diag}(y) + n$. Thus the only relevant inversions come from squares z satisfying $\mathrm{diag}(x) > \mathrm{diag}(z) > \mathrm{diag}(y)$ and $T(z) \in \{i, i+1\}$. We check directly that regardless of the value of $T(z)$, the cell z contributes exactly one inversion to $\mathrm{inv}_x(T) + \mathrm{inv}_y(T)$ and thus to $\mathrm{inv}_x(T') + \mathrm{inv}_y(T')$ as well. □

Lemma 2 combined with Lemma 1 shows that to prove that all ribbon functions are symmetric functions we only need to check it for horizontal ribbon strips λ/μ. For a horizontal ribbon strip λ/μ, let $I_{\lambda/\mu} \subset \mathbb{Z}$ be the set of diagonals such that $\mathrm{quot}_n(\lambda/\mu)$ contains a cell. It follows from Lemma 1 that the symmetry of $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ implies the symmetry for all horizontal strips ν/ρ with the same set of diagonals $I_{\nu/\rho} = I_{\lambda/\mu}$ – only the constant $e(\nu/\rho)$ has changed. It is easy to see that given a set of diagonals $I \subset \mathbb{Z}$, we can find a horizontal ribbon strip λ/μ such that $I_{\lambda/\mu} = I$ and so that λ/μ is tileable using vertical ribbons only. Thus the symmetry of all ribbon functions reduces to the symmetry of ribbon functions $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ corresponding to a horizontal ribbon strip λ/μ tileable only using vertical ribbons. In fact it is clear that we need only check this symmetry for such shapes which are connected.

4. Word sequence formulation of ribbon function symmetry

We now describe explicitly the bijection needed to prove symmetry of ribbon functions in terms of certain sequences. Let $n \geq 1$ be an integer.

Definition 3. A $(1,2,\emptyset)$-*word* is a sequence $(a_1, a_2, \ldots, a_m)$ where each $a_i \in \{1, 2, \emptyset\}$, such that whenever $a_i = 2$, then $a_{i+n} \neq 1$. The *form* F_a of a sequence $(a_1, a_2, \ldots, a_m)$ is the finite set $F_a = \{i \in [1,m] \mid a_i = \emptyset\}$. The weight $\mathrm{wt}(a)$ of such a word $a = (a_1, \ldots, a_m)$ is (μ_1, μ_2) where $\mu_i = \#\{j : a_j = i\}$.

Definition 4. A n-*local inversion* of a $(1,2,\emptyset)$-word $(a_1, a_2, \ldots, a_m)$ is a pair (i,j) satisfying $1 \leq i < j \leq m$ and $j - i < n$ such that $a_i = 2$ and $a_j = 1$. We let $\mathrm{linv}_n(w)$ denote the number of n-local inversions of w.

The following proposition makes the connection between $(1,2,\emptyset)$-words and a ribbon Bender Knuth involution.

Proposition 5. *The symmetry of ribbon functions is equivalent to the following identity on $(1,2,\emptyset)$-words for each positive integer m, form $F \subset [1,m]$ and weight (μ_1, μ_2):*

$$\sum_{a:wt(a)=(\mu_1,\mu_2)} q^{\mathrm{linv}_n(a)} = \sum_{a:wt(a)=(\mu_2,\mu_1)} q^{\mathrm{linv}_n(a)} \tag{1}$$

where the sum is over all $(1,2,\emptyset)$-words with length m, form F and specified weight.

Proof. We have already established that we need only be concerned with tableaux which are horizontal ribbon strips filled with ribbons labelled 1 and 2. Our $(1,2,\emptyset)$-words are simply the (reversed) reading words of these ribbon tableaux where the form F keeps track of the empty diagonals. The Proposition follows immediately from Lemma 1. □

We remark that when the form F is the emptyset, a bijection giving (1) is obtained by reversing the sequence and changing 2's to 1's and vice versa.

5. Connection with a result of van Leeuwen

Curiously, the symmetry of these special ribbon functions follows from a result of van Leeuwen concerning adding ribbons above and below a fixed lattice path. We identify the steps of an infinite lattice path P going up and right with a doubly infinite sequence $p = \{p_i\}_{i=-\infty}^{\infty}$ of 0's and 1's, where a 0 corresponds to a step to the right and a 1 corresponds to a step up. We may think of such lattice paths as the boundary of a shape (or partition) in which case the bit string is known as the edge sequence [8]. For our purposes, the indexing of $\{p_i\}$ is unimportant.

Van Leeuwen's result is the following [7, Claim 1.1.1].

Proposition 6. *Let $p = \{p_i\}_{i=-\infty}^{\infty}$ be a lattice path which is vertical at both ends. Let R_p denote the generating function*

$$R_p(X,q) = \sum_S q^{\mathrm{spin}(S)} X^{|S|}$$

where the sum is over all horizontal ribbon strips S that can be attached below p. Let $\tilde{p}$ denote p reversed. Then

$$R_p(X,q) = R_{\tilde{p}}(X,q).$$

Note that the generating functions $R_p(X,q)$ are finite, since only finitely many horizontal ribbon strips can be placed under a lattice path which is vertical at both ends. The lattice path $\tilde{p}$ should be thought of as rotating p upside-down, so that $R_{\tilde{p}}(X,q)$ enumerates the ways of adding a horizontal ribbon strip above p (see [7]).

We will also need the following technical lemma [7, Lemma 5.2.2] to make a calculation with spin. For a set $I \subset \mathbb{Z}$ of diagonals, we denote $\mathrm{spin}_I(T)$ to be the sum of the spins of the ribbons of T whose heads lie on the diagonals of I.

Lemma 7 ([7]). *Let λ, μ, ν be partitions so that λ/μ, λ/ν, μ/ν are all horizontal ribbon strips. Let $I, J \subset \mathbb{Z}$ be the set of diagonals occurring in λ/μ and μ/ν respectively. Then*

$$\mathrm{spin}_I(\lambda/\nu) - \mathrm{spin}(\lambda/\mu) = \mathrm{spin}_J(\lambda/\nu) - \mathrm{spin}(\mu/\nu).$$

Proposition 8. *Let λ/ν be a connected skew shape which is tileable by vertical ribbons only. Then $\mathcal{G}^{(n)}_{\lambda/\nu}(x_1, x_2; q)$ is a symmetric function.*

Proof. In the notation of Proposition 6, we pick p so that λ/ν is the shape obtained by adding as many vertical ribbons as possible below p to give a horizontal ribbon strip. Alternatively, we can think of λ/ν as the bounded region obtained by shifting the lattice path upwards n steps. Let $m = |\lambda/\nu|/n$. Let S_1 be a horizontal ribbon strip with $a \leq m$ ribbons added below p which we assume has shape μ/ν. Filling S_1 with 1's there is a unique way to add another horizontal ribbon strip S_2 filled with 2's to give a tableau $T \in \mathcal{R}^{a,b}_{\lambda/\nu}$.

Since $\mathrm{spin}_I(\lambda/\nu) = (n-1)|I|$ for any valid set of diagonals $I \subset I_{\lambda/\nu}$, we have $\mathrm{spin}(S_2) = (n-1)(2a-m) + \mathrm{spin}(S_1)$ by Lemma 7. Summing over all S_1, we get

$$\mathcal{G}^{(n)}_{\lambda/\nu}(x_1, x_2; q) = x_2^m q^{-(n-1)m} R_p\left(\frac{x_1}{x_2} q^{2(n-1)}, q^2\right).$$

However, we can also obtain the tableau T by counting the horizontal ribbon strip S_2 containing 2 first, so a similar argument gives $\mathcal{G}^{(n)}_{\lambda/\nu}(x_1, x_2; q) = x_1^m q^{-(n-1)m} R_{\tilde{p}}\left(\frac{x_2}{x_1} q^{2(n-1)}, q^2\right)$. Since $R_p = R_{\tilde{p}}$ by Proposition 6 we obtain $\mathcal{G}^{(n)}_{\lambda/\nu}(x_1, x_2; q) = \mathcal{G}^{(n)}_{\lambda/\nu}(x_2, x_1; q)$. □

The following theorem follows immediately from Proposition 8 and earlier discussion.

Theorem 9. *Let λ/μ be any skew shape tileable by n-ribbons. Then $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ is a symmetric function.*

Theorem 9 was first shown by Lascoux, Leclerc and Thibon [6] using an action of the Heisenberg algebra on the Fock space of $U_q(\widehat{\mathfrak{sl}}_n)$.

6. Generalised Yamanouchi domino tableaux

In this section we imitate a proof of the Littlewood Richardson rule due to Stembridge [10], which we apply to domino tableaux. We fix $n = 2$ throughout this section. Define the generalised (domino) q-Littlewood Richardson coefficients $c^{\lambda}_{\mu/\rho,\nu}(q)$ by

$$s_\nu(X)\mathcal{G}_{\mu/\rho}(X;q) = \sum_{\lambda} c^{\lambda}_{\mu/\rho,\nu}(q) s_\lambda(X).$$

Let $\{\sigma_r\}_{r=1}^{\infty}$ denote a set of fixed domino Bender-Knuth involutions which exist by Theorem 9. Let $w = w_1 w_2 \cdots w_k$ be a sequence of integers. Then the weight $\mathrm{wt}(w) = (\mathrm{wt}_1(w), \mathrm{wt}_2(w), \ldots)$ is the composition of k such that $\mathrm{wt}_i(w) = |\{j \mid w_j = i\}|$. If T is a ribbon tableau, let $T_{\geq j}$ and $T_{>j}$ denote the set of ribbons lying in diagonals which are $\geq j$ and $> j$ respectively (and similarly for $T_{<j}$ and $T_{\leq j}$). These are not tableaux, but the compositions $\mathrm{wt}(T_{\geq j})$ and $\mathrm{wt}(T_{>j})$ are well defined, in the usual manner.

Definition 10. Let λ be a partition. A word $w = w_1 w_2 \cdots w_k$ is λ-Yamanouchi if for any initial string $w_1 w_2 \cdots w_i$, and any integer l, we have $\mathrm{wt}_l(w_1 \cdots w_i) + \lambda_l \geq \mathrm{wt}_{l+1}(w_1 \cdots w_i) + \lambda_{l+1}$. A domino tableau D is λ-*Yamanouchi* if its reading word $\mathrm{r}(D)$ is λ-Yamanouchi.

One can check that (0)-Yamanouchi is essentially the notion of Yamanouchi introduced by Carré and Leclerc [1].

Theorem 11. *The generalised q-Littlewood Richardson coefficients are given by*

$$c^{\lambda}_{\mu/\rho,\nu}(q) = \sum_{Y} q^{\mathrm{spin}(Y)}$$

where the sum is over all ν-Yamanouchi domino tableaux Y of shape μ/ρ and weight $\lambda - \nu$.

Proof. Our proof will follow Stembridge's proof [10] nearly step by step. We will prove the Theorem in the variables $x_1, \ldots, x_m$ and will always think of a tableau D in terms of its 2-quotient. By definition,

$$\mathcal{G}_{\mu/\rho}(X;q) = \sum_{D} q^{\mathrm{spin}(D)} x^D$$

where the sum is over all semistandard domino tableaux of shape μ/ρ filled with numbers in $[1,m]$. Let $a_{\nu+\delta}$ denote the alternating sum $\sum_w (-1)^w x^{w(\nu+\delta)}$ where the sum is over all permutations $w \in S_m$. Then

$$a_{\nu+\delta}\mathcal{G}_{\mu/\rho}(X;q) = \sum_{w}\sum_{D} q^{\mathrm{spin}(D)}(-1)^w x^{D+w(\nu+\delta)} \tag{2}$$

$$= \sum_{D} q^{\mathrm{spin}(D)} \sum_{w} (-1)^w x^{w(D+\nu+\delta)} \tag{3}$$

$$= \sum_{D} q^{\mathrm{spin}(D)} a_{D+\nu+\delta}. \tag{4}$$

To obtain (3) we have used Theorem 9 to see that the weight generating function for domino tableaux with fixed spin is w invariant. We call D a Bad Guy if

$$\nu_k + \mathrm{wt}_k(D_{>j}) < \nu_{k+1} + \mathrm{wt}_{k+1}(D_{\geq j})$$

for some j and k. Of all such pairs (j,k), we pick one that maximises j and amongst those we pick the smallest k. Thus the reading word of $\mathrm{r}(D_{>j})$ is ν-Yamanouchi and the j-th diagonal of D contains a $k+1$ (and possibly a k) while the $(j+1)$-th diagonal contains no k.

Now let S be the set of dominoes obtained from $D_{<j}$ by including the k on the j-th diagonal if any. Set $S^* = \sigma_k(S)$. This makes sense since the squares of S containing a k or $k+1$ form a double horizontal strip which is actually of skew shape, so we can apply the Bender-Knuth involution. Now since $\mathrm{sh}(S) = \mathrm{sh}(S^*)$ we can attach S^* back onto $D_{\geq j}$ to obtain a tableau D^*. We check that D^* is a semistandard domino tableau. This is the case as only k's and $k+1$'s are changed into each other, and the boundary diagonals j and $j+1$ only contain $k+1$'s (there are two conditions to check, one for each tableau of the 2-quotient). Also note that if there is a k in diagonal j of S then there must be a $k+1$ immediately below it, so it will always remain a k in S^*.

It follows immediately from the construction that $D \mapsto D^*$ is an involution on the set of Bad Guys. We check that it is spin-preserving by counting the number of inversions. Since we have assumed that σ_k preserves spin, the only inversions that we have to be concerned about are those where $D(x) = k+1$ and $D(y) = k$ and $\mathrm{diag}(x) = j-1$ and $\mathrm{diag}(y) = j$. But if the j-th diagonal contains a k, then there is a $k+1$ immediately below it, so by Lemma 2, it can be ignored for calculations of spin in D, D^* and also S and S^*. So $\mathrm{spin}(D) = \mathrm{spin}(D^*)$.

Now,

$$a_{D+\nu+\delta} = -a_{D^*+\nu+\delta},$$

since $s_k(D+\nu+\delta) = D^* + \nu + \delta$, so the contributions of the Bad Guys to the sum (4) cancel out. The tableaux which are not Bad Guys are exactly the ν-Yamanouchi tableaux. Dividing both sides of (4) by a_δ and using the bialternant formula $s_\nu(X) = a_{\nu+\delta}/a_\delta$ now gives the Theorem. □

Unfortunately, this proof seems to fail for ribbon tableaux with $n > 2$. The similarly defined involution $T \mapsto T^*$ no longer preserves either semistandard-ness or spin.

We should remark also that Carré and Leclerc's algorithm mapping a domino tableau D to a pair (Y, T) of a Yamanouchi domino tableau and a usual Young tableau can also be interpreted in terms of the 2-quotient.

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