ON SJÖSTRAND'S SKEW SIGN-IMBALANCE IDENTITY

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Abstract. Recently, Sjöstrand gave an identity for the sign-imbalance of skew shapes. We give a quick proof of this using the skew domino Cauchy identity and some sign analysis for skew shapes.

1. The Theorem

Let T be a standard Young tableau with skew shape λ/μ. We will use the English notation for our tableaux, so that partitions are top-left justified. The *reading word* $r(T)$ of T is obtained by reading each row from left to right, starting with the top row. The *sign* $\mathrm{sign}(T)$ of T is the sign of $\mathrm{r}(T)$ as a permutation. The *sign-imbalance* $I_{\lambda/\mu}$ of λ/μ is given by

$$I_{\lambda/\mu} = \sum_T \mathrm{sign}(T),$$

where the summation is over all tableaux T with shape λ/μ.

For a partition λ, let $v(\lambda) = \sum_i \lambda_{2i}$ denote the sum of the even parts. denote the Generalizing an earlier conjecture of Stanley [7], Sjöstrand [6] proved the following identity.

Theorem 1 ([6]). *Let α be a partition and let $n \in \mathbb{N}$ be even. Then*

$$\sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}.$$

The aim of this note is to give a quick derivation of Theorem 1 using the techniques developed in [1] and the *skew domino Cauchy identity*. Let $\mathcal{G}_{\lambda/\mu}(X;q) = \sum_D q^{\mathrm{spin}(D)} x^{\mathrm{weight}(D)}$ be the spin-weight generating function of domino tableaux with shape λ/μ; see for example [1]. Here we will use the convention that $\mathrm{spin}(D)$ is equal to *half* the number of vertical dominoes in D. Though not stated explicitly, the following identity is a straightforward generalization of the "domino Cauchy identity" proved in any of [1, 2, 4].

Theorem 2. *Let α, β be two fixed partitions. Then*

$$\sum_\lambda \mathcal{G}_{\lambda/\alpha}(X;q)\mathcal{G}_{\lambda/\beta}(Y;q) = \prod_{i,j} \frac{1}{(1-x_iy_j)(1-qx_iy_j)} \sum_\mu \mathcal{G}_{\alpha/\mu}(X;q)\mathcal{G}_{\beta/\mu}(Y;q).$$

2. The Proof

Let D be a standard domino tableau with shape λ/μ. The sign $\mathrm{sign}(D)$ is equal to $\mathrm{sign}(T)$ where T is the standard Young tableau obtained from D, also with shape λ/μ, by replacing the domino labeled i by the numbers $2i-1$ and $2i$. The following result follows from a sign-reversing involution [1, 6, 7].

Lemma 3. *If λ/μ has an even number of squares, then its sign-imbalance is given by*

$$I_{\lambda/\mu} = \sum_D \mathrm{sign}(D),$$

where the summation is over all standard domino tableau with shape λ/μ.

Let $\delta \in D$ be a vertical domino occupying squares in rows $i-1$ and i. We call δ *nice* if the number of squares contained in λ/μ to the left of D in row i is odd. In other words, if δ lies in column j then it is nice if and only if $j - \mu_i$ is even. Let $\mathrm{nv}(D)$ denote the number of nice (and thus vertical) dominoes in D. Let $\mathrm{bv}(D)$ denote the number of non-nice vertical dominoes in D. Then we have $\mathrm{spin}(D) = \frac{\mathrm{nv}(D)+\mathrm{bv}(D)}{2}$.

Lemma 4. *Let D be a domino tableau of shape λ/μ. Then* $\mathrm{sign}(D) = (-1)^{\mathrm{nv}(D)}$.

Proof. This follows from the same argument as in the proof of [1, Proposition 21]. □

Lemma 5. *Let D be a domino tableau of shape λ/μ. The number* $\mathrm{nv}(D) - \mathrm{bv}(D)$ *depends only on the shape λ/μ.*

Proof. The number $\mathrm{nv}(D) - \mathrm{bv}(D)$ only depends on the tiling of λ/μ by dominoes. It is well known (see for example [5]) that every two such domino tilings can be connected by a number of moves of the form shown in Figure 1. These moves do not change the value of $\mathrm{nv}(D) - \mathrm{bv}(D)$. □

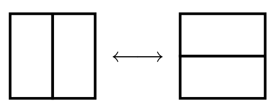

FIGURE 1. The "local move" which connects all domino tilings.

For each skew shape λ/μ, define $v'(\lambda/\mu) := \mathrm{nv}(D) - \mathrm{bv}(D)$ where D is any domino tableau with shape λ/μ.

Lemma 6. *We have* $v(\lambda) + v(\mu) \equiv v'(\lambda/\mu) \mod 2$.

Proof. This is straight forward to prove by induction on the size of λ, starting with $\lambda = \mu$ and adding dominoes. □

Proposition 7. *Let λ/μ have an even number of squares. Then*

$$I^2_{\lambda/\mu} = (-1)^{v(\lambda/\mu)} \left(\sum_D (-1)^{\mathrm{spin}(D)} \right)^2,$$

where the summation is over all domino tableaux of shape λ/μ.

Proof. We have

$$\begin{aligned}
I_{\lambda/\mu}^2 &= \left(\sum_D \mathrm{sign}(D)\right)^2 && \text{by Lemma 3} \\
&= \left(\sum_D (-1)^{\mathrm{nv}(D)}\right)^2 && \text{by Lemma 4,} \\
&= \left(\sum_D (-1)^{\mathrm{spin}(D)+v'(\lambda/\mu)/2}\right)^2 && \text{using } \mathrm{spin}(D) = \tfrac{\mathrm{nv}(D)+\mathrm{bv}(D)}{2}, \\
&= (-1)^{v(\lambda/\mu)} \left(\sum_D (-1)^{\mathrm{spin}(D)}\right)^2 && \text{by Lemma 6.}
\end{aligned}$$

□

Proof of Theorem. In Theorem 2, let $\alpha = \beta$ and $q = -1$. Then take the coefficient of $x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n$ on both sides. Using Proposition 7, this gives exactly Theorem 1. □

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