Combinatorics of Ribbon Tableaux

by

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Abstract

This thesis begins with the study of a class of symmetric functions $\{\mathcal{G}_\lambda\}$ which are generating functions for ribbon tableaux (hereon called *ribbon functions*), first defined by Lascoux, Leclerc and Thibon. Following work of Fomin and Greene, I introduce a set of operators called *ribbon Schur operators* on the space of partitions. I develop the theory of ribbon functions using these operators in an elementary manner. In particular, I deduce their symmetry and recover a theorem of Kashiwara, Miwa and Stern concerning the Fock space $\mathbf{F}$ of the quantum affine algebras $U_q(\widehat{\mathfrak{sl}}_n)$.

Using these results, I study the functions $\mathcal{G}_\lambda$ in analogy with Schur functions, giving:

- a Pieri and dual-Pieri formula for ribbon functions,
- a ribbon Murnaghan-Nakayama formula,
- ribbon Cauchy and dual Cauchy identities,
- and a $\mathbb{C}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ which sends each $\mathcal{G}_\lambda$ to $\mathcal{G}_{\lambda'}$.

The study of the functions $\mathcal{G}_\lambda$ will be connected to the Fock space representation $\mathbf{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ via a linear map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ which sends the standard basis of $\mathbf{F}$ to the ribbon functions. Kashiwara, Miwa and Stern [28] have shown that a copy of the Heisenberg algebra H acts on $\mathbf{F}$ commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$. Identifying the Fock Space of H with the ring of symmetric functions $\Lambda(q)$ I will show that $\mathbf{\Phi}$ is in fact a map of H-modules with remarkable properties.

In the second part of the thesis, I give a combinatorial generalisation of the classical Boson-Fermion correspondence and explain how the map $\mathbf{\Phi}$ is an example of this more general phenomena. I show how certain properties of many families of symmetric functions arise naturally from representations of Heisenberg algebras. The main properties I consider are a tableaux-like definition, a Pieri-style rule and a Cauchy-style identity. Families of symmetric functions which can be viewed in this manner include Schur functions, Hall-Littlewood functions, Macdonald polynomials and the ribbon functions. Using work of Kashiwara, Miwa, Petersen and Yung, I define generalised ribbon functions for certain affine root systems Φ of classical type. I prove a theorem relating these generalised ribbon functions to a speculative global basis of level 1 q-deformed Fock spaces.

Thesis Supervisor: Richard P. Stanley
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Contents

List of Figures

Chapter 1

Introduction

The study of symmetric functions has grown enormously since Girard and Newton first studied the subject in the seventeenth century. In the last century, symmetric function theory has developed at the crossroads of combinatorics, algebraic geometry and representation theory. The ring of symmetric functions $\Lambda = \Lambda_{\mathbb{Q}}$ contains a distinguished basis $\{s_\lambda\}$ known as the *Schur functions*. Schur functions simultaneously represent the characters of the symmetric group S_n, the characters of the general linear group $GL(N)$, the Schubert classes of the Grassmannian Gr_{kn} and the weight generating functions of Young tableaux.

In recent years, there has been an explosion of interest in q- and q,t-analogues of symmetric functions; see for example [46, 36, 53, 62]. With each such family of symmetric functions, the initial aim is both to connect the functions with representation theory, algebraic geometry or some other field; and to generalise the numerous properties of Schur functions to the new family of symmetric functions. The first part of this thesis is concerned with the latter task for a family of symmetric functions which we call *ribbon functions*, defined by Lascoux, Leclerc and Thibon [38]. The second part of this thesis tries to explain the results of the first part in the wider context of representations of Heisenberg algebras and the Boson-Fermion correspondence.

Ribbon functions are defined combinatorially as the spin-weight generating functions of ribbon tableaux:

$$\mathcal{G}_\lambda^{(n)}(X;q) = \sum_T q^{s(T)} x^{w(T)}$$

where the sum is over all *semistandard n-ribbon tableaux* (see Figure 1-1) of shape λ, and $s(T)$ and $w(T)$ are the spin and weight of T respectively. The *spin* of a ribbon is the number of rows in the ribbon, minus 1. The definition of a semistandard ribbon tableau is analagous to the definition of a semistandard Young tableau, with boxes replaced by ribbons (or border strips) of length n. When $n = 1$, ribbon functions reduce to the Schur functions. When $q = 1$, we obtain products of n Schur functions. The definition of ribbon functions can be extended naturally to skew shapes λ/μ.

To prove that the functions $\mathcal{G}_\lambda^{(n)}(X;q)$ were symmetric Lascoux, Leclerc and Thibon connected them to the (level 1) Fock space representation $\mathbf{F}$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. The crucial property of $\mathbf{F}$ is that it affords an action of a Heisenberg algebra H, commuting with the action of $U_q(\widehat{\mathfrak{sl}}_n)$, discovered by Kashiwara, Miwa and Stern [28]. In particular, they showed that as a $U'_q(\widehat{\mathfrak{sl}}_n) \times H$-module, $\mathbf{F}$ decomposes as

$$\mathbf{F} \cong V_{\Lambda_0} \otimes \mathbb{Q}(q)[H_-]$$

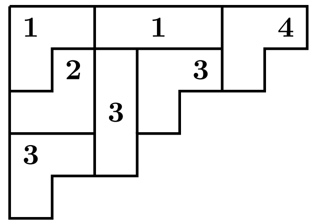

Figure 1-1: A 3-ribbon tableau with shape $(7, 6, 4, 3, 1)$, weight $(2, 1, 3, 1)$ and spin 7.

where V_{Λ_0} is the highest weight representation of $U'_q(\widehat{\mathfrak{sl}}_n)$ with highest weight Λ_0 and $\mathbb{Q}(q)[H_-]$ is the usual Fock space representation of the Heisenberg algebra.

The initial investigations of ribbon functions were focused on the q-Littlewood Richardson coefficients $c^\mu_\lambda(q)$ of the expansion of $\mathcal{G}^{(n)}_\lambda(X;q)$ in the Schur basis:

$$\mathcal{G}^{(n)}_\lambda(X;q) = \sum_\mu c^\mu_\lambda(q) s_\mu(X).$$

These are q-analogues of Littlewood Richardson coefficients. Leclerc and Thibon [40] showed that the polynomials $c^\mu_\lambda(q)$ are coefficients of global bases of the Fock Space $\mathbf{F}$. Results of Varagnolo and Vasserot [60] then imply that they are parabolic Kazhdan-Lusztig polynomials of type A. Finally, geometric results of Kashiwara and Tanisaki [29] show that they are polynomials in q with non-negative coefficients. Much interest has also developed in connecting ribbon tableaux and the q-Littlewood Richardson coefficients to rigged configurations and the generalised Kostka polynomials defined by Kirillov and Shimozono [30], Shimozono and Weyman [52] and Schilling and Warnaar [50].

In a mysterious development, Haglund et. al. [17] have conjectured connections between diagonal harmonics and ribbon functions. More recently, Haglund, Haiman and Loehr [16] have found an expression for Macdonald polynomials in terms of the skew ribbon functions $\mathcal{G}^{(n)}_{\lambda/\mu}$. Unfortunately, the positivity of the skew q-Littlewood Richardson coefficients does not follow from the representation theory, so a proof of the Macdonald positivity conjecture is yet to result from this approach.

The *Fock space* $\mathbf{F}$ can be viewed as the vector space over $\mathbb{Q}(q)$ spanned by partitions with a natural inner product $\langle .,. \rangle : \mathbf{F} \times \mathbf{F} \to \mathbb{Q}(q)$ given by $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$. Our study of ribbon functions begins in Chapter 3 with the definition of combinatorial operators $\{u^{(n)}_i \mid i \in \mathbb{Z}\}$ called *ribbon Schur operators* on $\mathbf{F}$:

$$u^{(n)}_i : \lambda \longmapsto \begin{cases} q^{\mathrm{spin}(\mu/\lambda)}\mu & \text{if } \mu/\lambda \text{ is a } n\text{-ribbon with head lying on the } i\text{-th diagonal,} \\ 0 & \text{otherwise.} \end{cases}$$

Following work of Fomin and Greene [12], many properties of ribbon functions can be phrased in terms of ribbon Schur operators. In particular, we identify a commutative subalgebra $\Lambda(\mathbf{u}) \subset \mathbb{Q}(q)[\ldots, u_{-1}, u_0, u_1, \ldots]$ abstractly isomorphic to the ring of symmetric functions. This algebra should be thought of as the (Hopf)-dual of the symmetric function algebra in which ribbon functions are defined. We give explicit formulae for certain "non-commutative Schur functions" $s_\lambda(\mathbf{u}) \in \Lambda(\mathbf{u})$ which are related to ribbon functions by the formula

$$\langle s_\lambda(\mathbf{u}) \cdot \nu, \mu \rangle = c^\lambda_{\nu/\mu}(q).$$

Our results thus imply some new positivity results for skew q-Littlewood Richardson coefficients.

Making computations involving $\Lambda(\mathbf{u})$ and the adjoint algebra $\Lambda^{\perp}(\mathbf{u}) \subset \mathrm{End}_{\mathbb{Q}(q)}(\mathbf{F})$, we obtain a non-commutative "Cauchy identity" for the operators u_i. In this way we recover using just linear algebra and combinatorics of ribbons the action of the Heisenberg algebra on $\mathbf{F}$ due to Kashiwara, Miwa and Stern mentioned above. In particular, we obtain an elementary proof of the symmetry of ribbon functions.

In Chapter 4, we use the action of the Heisenberg algebra on $\mathbf{F}$ to deduce properties of ribbon functions in analogy with Schur functions:

- A ribbon Pieri formula (Theorem 4.12):

$$h_k[\left(1+q^2+\cdots+q^{2(n-1)}\right)X]\mathcal{G}_\nu(X;q) = \sum_{\mu} q^{s(\mu/\nu)}\mathcal{G}_\mu(X;q).$$

  where the sum is over all μ such that μ/ν is a *horizontal ribbon strip* of size k. The notation $h_k[\left(1+q^2+\cdots+q^{2(n-1)}\right)X]$ denotes a plethysm.

- A ribbon Murnaghan-Nakayama-rule (Theorem 4.22):

$$\left(1+q^{2k}+\cdots+q^{2k(n-1)}\right)p_k\mathcal{G}_\nu(X;q) = \sum_{\mu}\mathcal{X}^k_{\mu/\nu}(q)\mathcal{G}_\mu(X;q).$$

  where $\mathcal{X}^k_{\mu/\nu}(q)$ can be expressed as an alternating sum of spins over certain "border n-ribbon strips" of size k.

- A ribbon Cauchy (and dual Cauchy) identity (Theorem 4.28):

$$\sum_{\lambda}\mathcal{G}_{\lambda/\delta}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q) = \prod_{i,j}\prod_{k=0}^{n-1}\frac{1}{1-x_iy_jq^{2k}}$$

  where the sum is over all partitions λ with a fixed n-core δ. A combinatorial proof of this was given recently by van Leeuwen [42].

- A $\mathbb{Q}$-algebra isomorphism $\omega_n : \Lambda(q) \to \Lambda(q)$ (Theorem 4.26) satisfying

$$\omega_n(\mathcal{G}_\lambda(X;q)) = \mathcal{G}_{\lambda'}(X;q).$$

  Even the existence of a linear map with such a property is not obvious as the functions $\mathcal{G}_\lambda$ are not linearly independent.

One should expect these formulae to be important properties. For example, the Pieri formula for Schur functions calculates the intersection of an arbitrary Schubert variety with a special Schubert variety in the Grassmannian. The Murnaghan-Nakayama rule calculates the irreducible characters of the symmetric group.

The connection between ribbon functions and the action of the Heisenberg algebra is made explicit by showing that the map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ defined by

$$\lambda \longmapsto \mathcal{G}_\lambda \tag{1.1}$$

is a map of H-modules, after identifying $\mathbb{Q}(q)[H_-]$ with the ring of symmetric functions $\Lambda(q)$ in the usual way. The map $\mathbf{\Phi}$ has the further remarkable property that it changes certain linear maps into algebra maps, as follows.

Lascoux, Leclerc and Thibon [37] have constructed a global basis of $\mathbf{F}$ which extends Kashiwara's global crystal basis of V_{Λ_0}. They used a bar involution $^- : \mathbf{F} \to \mathbf{F}$ which extends Kashiwara's involution on V_{Λ_0}. Another semi-linear involution, denoted $v \mapsto v'$ was also introduced and further studied in [40] which satisfied the property $\langle \overline{u}, v \rangle = \langle u', \overline{v'} \rangle$ for $u, v \in \mathbf{F}$. We shall see that if we restrict $\mathbf{\Phi}$ to the space of highest weight vectors of $\mathbf{F}$ for the $U_q(\widehat{\mathfrak{sl}}_n)$ action, then both involutions become algebra isomorphisms under the map $\mathbf{\Phi}$. In particular the "image" of the involution $v \mapsto v'$ is simply ω_n.

In Chapter 5, we explain how (1.1) should be thought of as a generalisation of the *Boson-Fermion correspondence*. The classical Boson-Fermion correspondence identifies the image of semi-infinite wedges in the Fermionic Fock space as the Schur functions in the Bosonic Fock space. Our investigations are motivated by the observation that many classical families of symmetric functions possess a trio of properties: a combinatorial tableaux-like definition, a Pieri-style rule and a Cauchy-style identity. Such families of symmetric functions include Schur functions, Hall Littlewood functions, Macdonald polynomials and ribbon functions. We shall explain these phenomena using representations of Heisenberg algebras.

Our aim is to generalise the Boson-Fermion correspondence to any representation of a Heisenberg algebra H with "parameters" a_i. A Heisenberg algebra is generated by $\{B_k : k \in \mathbb{Z}\backslash\{0\}\}$ satisfying

$$[B_k, B_l] = l \cdot a_l \cdot \delta_{k,-l}$$

for some non-zero parameters a_l satisfying $a_l = -a_{-l}$. Given a representation V of H with a distinguished basis $\{v_s \mid s \in S\}$ for some indexing set S and a highest weight vector in V, we define two families $\{F_s \mid s \in S\}$ and $\{G_s \mid s \in S\}$ of symmetric functions. These definitions are combinatorial and tableaux-like in the sense that they give the expansion of F_s or G_s in terms of monomials. The map $v_s \mapsto G_s$ turns out to be a map of H-modules for a suitable action of H on Λ. The sum $\sum_s F_s(X)G_s(Y)$ satisfies a Cauchy identity: it has an explicit product formula involving the parameters a_i. Lastly, we find symmetric functions $h_k[a_i] \in \Lambda$ so that both $h_k[a_i]F_s$ and $h_k[a_i]G_s$ have Pieri-like expressions.

There is another part of the Boson-Fermion correspondence involving *vertex operators* which we have chosen to largely ignore. This point of view has been studied by Jing and Macdonald [19, 20, 46].

In the last part of the thesis, we use our generalisation of the Boson-Fermion correspondence to define ribbon functions for other Fock space representations. Our definition uses a construction of the Fock space representations for quantum affine algebras due to Kashiwara, Miwa, Petersen and Yung [27]. They define Fock space representations $\mathcal{F}$ for the quantum affine algebras of types $A^{(2)}_{2n}$, $B^{(2)}_n$, $A^{(2)}_{2n-1}$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$. The main theorem of [27], for our purposes, is that $\mathcal{F}$ decomposes as $V_\lambda \otimes \mathbb{Q}[H_-]$ as a $U'_q(\mathfrak{g}) \otimes H$ module. Another construction of $\mathcal{F}$ is given by Kang and Kwon [24] in terms of *Young walls*, though they do not consider the action of the Heisenberg algebra.

The construction of these Fock spaces in [27] relies on a choice of a (level 1) perfect crystal B for the quantum affine algebra. The Fock space representation is indexed by certain "normally-ordered" elements $b_1 \otimes b_2 \otimes \cdots$ in a semi-infinite tensor product $B \otimes B \otimes \cdots$ of this crystal. In the notation above, this will be our indexing set S. The action of a Heisenberg algebra on the Fock space is also given explicitly in [27], and we use it to define generalised ribbon functions $F_s^\Phi \in \Lambda(q)$. In the case $\Phi = A_n^{(1)}$, we explain how one recovers Lascoux-Leclerc-Thibon ribbon functions. We also give examples of ribbons and ribbon functions for $\Phi = A_{2n}^{(2)}$.

These generalised ribbon functions are likely to be interesting from both the combinatorial and representation theoretic points of view, though the calculation of ribbon functions is considerably harder. We generalise a result of Leclerc and Thibon to show that the generalised q-Littlewood Richardson coefficients $c_s^{\lambda,\Phi} \in \mathbb{Q}(q)$ given by $F_s^\Phi = \sum_\lambda c_s^{\lambda,\Phi} s_\lambda$ are also coefficients of a speculative "global basis" of $\mathcal{F}$.

Chapter 2

Ribbon tableaux and ribbon functions

In this chapter we give the definitions for tableaux and symmetric functions needed throughout this thesis, and define the ribbon tableaux generating functions of Lascoux, Leclerc and Thibon.

2.1 Ribbon tableaux

2.1.1 Partitions and Tableaux

A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a list of non-increasing integers. We will call l the *length* of λ, and denote it by $l(\lambda)$. We will say that λ is a partition of $\lambda_1+\lambda_2+\ldots+\lambda_l = |\lambda|$ and write $\lambda \vdash |\lambda|$. A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is an ordered list of non-negative integers. As above, we will say that α is a composition of $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. We use the usual notation concerning partitions and do not distinguish between a partition and its Young diagram. Let $m_k(\lambda)$ denote the number of parts of λ equal to k. Let λ' denote the partition *conjugate* to λ, given by $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$. We shall write $\mathcal{P}$ for the set of partitions. We shall always draw our partitions in the English notation, so that the parts are top left justified.

The *edge sequence* $p(\lambda) = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$ of a partitions λ is the doubly infinite bit sequence obtained by drawing the partition in the English notation and reading the "edge" of the partition from bottom left to top right − writing a 1 if you go up and writing a 0 if you go to the right (see Figure 2-1). We shall normalise our notation for edge sequences by requiring that the empty partition $\emptyset$ has edge sequence $p(\emptyset)_i = 1$ for $i \leq 0$ and $p(\emptyset)_i = 0$ for $i \geq 1$. Adding a box to a partition corresponds to changing two adjacent entries of the edge sequence (p_i, p_{i+1}) from $(0, 1)$ to $(1, 0)$.

If λ and μ are partitions we say that λ contains μ if $\lambda_i \geq \mu_i$ for each i, and write $\mu \subset \lambda$. A *skew shape* is a pair of partitions λ, μ such that $\mu \subset \lambda$, denoted λ/μ. The skew shape λ/μ is a *horizontal strip* if it contains at most one square in each column. A skew shape λ/μ is a *border strip* if it is connected, and does not contain any 2×2 square. The *height* $h(b) \in \mathbb{N}$ of a border strip b is the number of rows in it, minus 1. A *border strip tableau* T of shape λ/μ is a chain of partitions

$$\mu = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

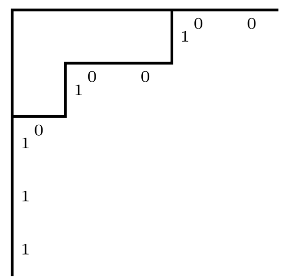

Figure 2-1: The edge sequence $p(31) = (\ldots, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, \ldots)$.

such that each μ^{i+1}/μ^i is a border strip. The height of a *border strip tableau* T is the sum of the heights of its border strips.

A *semistandard* tableau T of shape λ/μ is a filling of each square $(i, j) \in \lambda/\mu$ with a positive integer such that the rows are non-decreasing and the columns are increasing. We write $\mathrm{sh}(T) = \lambda/\mu$. The weight $w(T)$ of such a tableau T is the composition α such that α_i is the number of occurrences of i in T. The tableau is *standard* if the numbers which occur are exactly those of $[m] = \{1, 2, \ldots, m\}$ for some integer m. A semistandard tableau of shape λ/μ can be thought of as a chain of partitions $\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \lambda$ such that each λ^i/λ^{i-1} is a horizontal strip.

2.1.2 Cores and ribbons

Let $n \geq 1$ be a positive integer. When a border strip has n squares for the distinguished integer n, we will call it a *n-ribbon* or just a *ribbon*. The height of the ribbon r will then be called its *spin* $\mathrm{spin}(r)$. The reader should be cautioned that in the literature the spin is often defined as half of this. The *head* of a ribbon is its top-right most square.

Let λ be a partition. Its *n-core* $\tilde{\lambda}$ is obtained from λ by removing n-ribbons until it is no longer possible to do so, and does not depend on how the ribbons are removed. Adding a n-ribbon to a partition λ corresponds to finding an index $i \in \mathbb{Z}$ such that $p_i(\lambda) = 1$ and $p_{i+n}(\lambda) = 0$, then changing those two bits to $p_i(\lambda) = 0$ and $p_{i+n}(\lambda) = 1$.

Let λ be an n-core with edge sequence $p(\lambda) = (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$. Then there is no index i so that $p_i(\lambda) = 0$ and $p_{i+n}(\lambda) = 1$. Equivalently, the subsequences

$$p^{(i)}(\lambda) = (\ldots, p_{i-2n}, p_{i-n}, p_i, p_{i+n}, p_{i+2n}, \ldots)$$

all look like $(\ldots, 1, 1, 1, 1, 0, 0, 0, 0, \ldots)$ with a suitable shift. Define the *offset sequence* $(d_0, d_1, \ldots, d_{n-1}) \in \mathbb{Z}^n$ by requiring that $p_{i+nd_i} = 0$ and $p_{i+n(d_i-1)} = 1$. The offsets satisfy $d_{i-n} = d_i + 1$ and $d_1 + d_2 + \cdots + d_n = 0$ and completely determine the n-core.

The *n-quotient* of λ a n-tuple of partitions denoted by $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ and are defined by requiring that $p_i(\lambda^{(j)}) = p_{(i+d_j)n+j}(\lambda)$ where $d_j = d_j(\tilde{\lambda})$. The n-quotient and n-core of λ completely define λ. They satisfy $|\lambda| = |\tilde{\lambda}| + n(|\lambda^{(0)}| + \cdots + |\lambda^{(n-1)}|)$.

Let $\mathcal{P}_\delta$ to denote the set of partitions λ such that $\tilde{\lambda} = \delta$ for an n-core $\delta = \tilde{\delta}$. A *ribbon tableau* T of shape λ/μ is a tiling of λ/μ by n-ribbons and a filling of each ribbon with a positive integer (see Figure 1-1). If these numbers are exactly those of $[m]$, for some m, then the tableau is called *standard*. We will use the convention that a ribbon tableau of shape λ where $\tilde{\lambda} \neq \emptyset$ is simply a ribbon tableau of shape $\lambda/\tilde{\lambda}$. A ribbon tableau is *semistandard* if for each i

1. removing all ribbons labelled j for $j > i$ gives a valid skew shape $\lambda_{\leq i}/\mu$ and,

2. the subtableau containing only the ribbons labelled i form a *horizontal n-ribbon strip.*

A tiling of a skew shape λ/μ by n-ribbons is a *horizontal ribbon strip* if the topright-most square of every ribbon touches the northern edge of the shape (see Figure 2-2). Abusing notation, we will also call the skew shape λ/μ a horizontal ribbon strip λ/μ if such a tiling exists (which is necessarily unique). Similarly, one has the notion of a *vertical ribbon strip.* If λ/μ is a horizontal ribbon strip then λ'/μ' is a vertical ribbon strip. If λ/μ has k ribbons then we have $\mathrm{spin}(\lambda/\mu) + \mathrm{spin}(\lambda'/\mu') = k(n-1)$, where in one case we are taking the spin of the tiling as a horizontal ribbon strip and in the second case we are taking the spin of the tiling as a vertical ribbon strip.

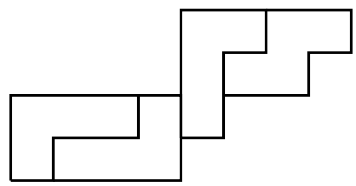

Figure 2-2: A horizontal 4-ribbon strip with spin 5.

We will often think of a ribbon tableau as a chain of partitions

$$\tilde{\lambda} = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r = \lambda$$

where each μ^{i+1}/μ^i is a horizontal ribbon strip. The *spin* $\mathrm{spin}(T)$ of a ribbon tableau T is the sum of the spins of its ribbons. If λ/μ is a horizontal ribbon strip then $\mathrm{spin}(\lambda/\mu)$ denotes the spin of the unique tiling of λ/μ such that the topright-most square of every ribbon touches the northern edge of the shape. The *weight* $w(T)$ of a tableau is the composition counting the occurrences of each value in T.

Littlewood's *n-quotient map* ([43], see also [57]) gives a weight preserving bijection between semistandard ribbon tableaux T of shape λ and n-tuples of semistandard Young tableau $\{T^{(0)}, \ldots, T^{(n-1)}\}$ of shapes $\{\lambda^{(0)}, \ldots, \lambda^{(n-1)}\}$ respectively. The n-quotient map for tableaux can be defined by treating a tableau as a chain of partitions and applying the n-quotient operation for each partition in the chain. Abusing language, we shall also refer to $\{T^{(0)}, \ldots, T^{(n-1)}\}$ as the n-quotient of T. Schilling, Shimozono and White [51] and separately Haglund et. al. [17] have described the spin of a ribbon tableau in terms of an inversion number of the n-quotient. Note that a shape λ/μ is a horizontal ribbon strip if and only if its n-quotient is a n-tuple of horizontal strips.

2.2 Ribbon functions

2.2.1 Symmetric functions

We review some standard notation and results in symmetric function theory (see [46] for details).

Let $\Lambda = \Lambda_{\mathbb{Q}}$ denote the *ring of symmetric functions* over $\mathbb{Q}$. We will write $\Lambda(q)$ for the ring of symmetric functions over $\mathbb{Q}(q)$. If K is any field of characteristic 0, we write Λ_K for the algebra of symmetric functions over K. It is well known that the *Schur functions* s_λ are orthogonal with respect to the *Hall inner product* $\langle .,. \rangle$ on Λ. If $f \in \Lambda$ then $f^\perp$ denotes the adjoint to multiplication by f with respect to $\langle .,. \rangle$, so that $\langle fg, h \rangle = \langle g, f^\perp \cdot h \rangle$. We will denote the *homogeneous, elementary, monomial* and *power sum* symmetric functions by h_λ, e_λ, m_λ and p_λ respectively. Recall that we have $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ where $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$. Each of the sets $\{p_i\}$, $\{e_i\}$ and $\{h_i\}$ generate Λ. We will write X to mean $(x_1, x_2, \ldots)$. Thus $s_\lambda(X) = s_\lambda(x_1, x_2, \ldots)$.

The Schur functions can be defined combinatorially in terms of Young tableaux:

$$s_\lambda = \sum_T x^{w(T)}$$

where the sum is over all semistandard tableaux T of shape λ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. Recall that the *Kostka numbers* $K_{\lambda\mu}$ are defined by $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$. Thus $K_{\lambda\mu}$ is equal to the number of semistandard tableaux of shape λ and weight μ.

The ring of symmetric functions has an algebra involution $\omega : \Lambda \to \Lambda$ defined by $\omega(h_i) = e_i$. It satisfies $\omega(s_\lambda) = s_{\lambda'}$ and is an isometry with respect to $\langle .,. \rangle$. The *Cauchy kernel* $\Omega(X, Y) := \prod_{i,j} \frac{1}{1 - x_i y_j}$ satisfies

$$\Omega(X, Y) = \sum_\lambda h_\lambda(X) m_\lambda(Y) = \sum_\lambda s_\lambda(X) s_\lambda(Y) = \sum_\lambda z_\lambda^{-1} p_\lambda(X) p_\lambda(Y).$$

The *Pieri rule* allows one to calculate the product of a Schur function by a homogeneous symmetric function:

$$h_\lambda s_\lambda = \sum_\mu s_\mu$$

where the sum is over all μ such that μ/λ is a horizontal strip. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The Jacobi-Trudi formula expresses Schur function s_λ as a determinant of homogeneous symmetric functions:

$$s_\lambda = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-2} & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-3} & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & \cdots & h_{\lambda_k-1} & h_{\lambda_k} \end{pmatrix}.$$

Here we take $h_0 = 1$ and $h_i = 0$ for $i < 0$.

Let $f \in \Lambda$. We recall the definition of the *plethysm* $g \mapsto g[f]$. Write $g = \sum_\lambda c_\lambda p_\lambda$. Then we have

$$g[f] = \sum_\lambda c_\lambda \prod_{i=1}^{l(\lambda)} f(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots).$$

Thus the plethysm by f is the (unique) algebra endomorphism of Λ which sends $p_k \mapsto f(x_1^k, x_2^k, \ldots)$. When $f(x_1, x_2, \ldots; q) \in \Lambda(q)$ for a distinguished element q, we define the plethysm as $p_k \mapsto f(x_1^k, x_2^k, \ldots; q^k)$. Note that plethysm does not commute with specialising q to a complex number.

For example, the plethysm by $(1+q)p_1$ is given by sending $p_k \mapsto (1+q^k)p_k$ and extending

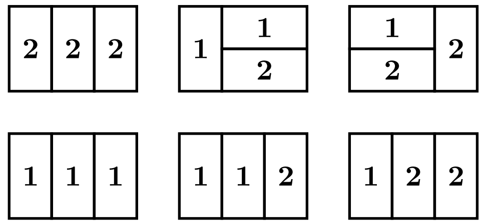

Figure 2-3: The semistandard domino tableaux contributing to $\mathcal{G}^{(2)}_{(3,3)}(x_1, x_2; q)$.

to an algebra isomorphism $\Lambda(q) \to \Lambda(q)$. In such situations we will write $f[(1+q)X]$ for $f[(1+q)p_1]$.

2.2.2 Lascoux, Leclerc and Thibon's ribbon functions

We now define the central objects of this paper as introduced by Lascoux, Leclerc and Thibon. An integer $n \geq 1$ is fixed throughout and will often be suppressed in the notation.

Definition 2.1 ([38]). Let λ/μ be a skew partition, tileable by n-ribbons. Define the symmetric functions $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q) = \mathcal{G}_{\lambda/\mu}(X;q) \in \Lambda(q)$ as:

$$\mathcal{G}_{\lambda/\mu}(X;q) = \sum_T q^{\mathrm{spin}(T)} x^{w(T)}$$

where the sum is over all semistandard ribbon tableaux T of shape λ/μ. These functions will be loosely called *ribbon functions*.

When $\mu = \emptyset$ we will write $\mathcal{G}_\lambda(X;q)$ in place of $\mathcal{G}_{\lambda/\emptyset}(X;q)$. The fact that the functions $\mathcal{G}_{\lambda/\mu}(X;q)$ are symmetric is not obvious from the combinatorial definition, and was first shown by in [38] using representation theoretic results in [28]. We shall give an elementary proof of the symmetry in Theorem 3.12.

Example 2.2. Let $n = 2$ and $\lambda = (3,3)$. Then we have

$$\mathcal{G}^{(2)}_{(3,3)}(x_1, x_2; q) = q^3(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) + q(x_1^2 x_2 + x_2^2 x_1),$$

corresponding to the domino tableaux in Figure 2-3. In fact,

$$\mathcal{G}^{(2)}_{(3,3)}(X;q) = q s_{2,1}(X) + q^3 s_3(X).$$

The symmetry of $\mathcal{G}^{(2)}_{(3,3)}(X;q)$ is already non-obvious. Note also that $\mathcal{G}^{(2)}_{(3,3)}(X;1) = s_1 s_2$.

Let λ/μ be a skew shape tileable by n-ribbons. Then define

$$\mathcal{K}_{\lambda/\mu,\alpha}(q) = \sum_T q^{\mathrm{spin}(T)},$$

the spin generating function of all semistandard ribbon tableaux T of shape λ/μ and weight α. Thus $\mathcal{G}_{\lambda/\mu}(X;q) = \sum_\alpha \mathcal{K}_{\lambda/\mu,\alpha}(q) x^\alpha$. Also define the *$q$-Littlewood Richardson coefficients* $c^\nu_{\lambda/\mu}(q)$ by

$$\mathcal{G}_{\lambda/\mu}(X;q) := \sum_\nu c^\nu_{\lambda/\mu}(q) s_\nu(X).$$

When $q = 1$, the ribbon functions become products of Schur functions (see [38]):

$$\mathcal{G}_{\lambda/\tilde{\lambda}}(X;1) = s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}. \tag{2.1}$$

This is a consequence of Littlewood's n-quotient map. In fact, up to sign, $\mathcal{G}_\lambda(X;1)$ is essentially $\phi_n(s_\lambda)$ where ϕ_n is the adjoint operator to taking the plethysm by p_n ([38]). More generally, $\mathcal{G}_{\lambda/\mu}(X;q)$ reduces to a product of skew Schur functions at $q = 1$.

When $n = 1$, we have $\mathcal{G}_\lambda(X;q) = s_\lambda(X)$ and we just obtain the usual Schur functions. One of the main aims of this thesis is to generalise some of the properties of Schur functions described in Section 2.2.1 to arbitrary ribbon functions.

Remark 2.3. In [38], another set of symmetric functions $\mathcal{H}_\lambda(X;q)$ defined by $\mathcal{H}_\lambda(X;q) = \mathcal{G}_{n\lambda}(X;q)$ is studied. It is not hard to see that $\mathcal{H}_\lambda(X;1) = s_\lambda(X) + \sum_{\mu \prec \lambda} d_{\lambda,\mu} s_\mu(X)$ for some $d_{\lambda,\mu} \in \mathbb{Z}$ where $\prec$ denotes the usual dominance order on partitions. Thus the functions $\mathcal{H}_\lambda(X;q)$ form a basis of $\Lambda(q)$ over $\mathbb{Q}(q)$. In [38] it is shown that the "cospin" version $\tilde{\mathcal{H}}_\lambda(X;q)$ generalise the modified Hall-Littlewood functions $Q'_\lambda(X;q)$; see [46].

Chapter 3

Ribbon Schur operators

This Chapter contains material from the paper [35], with some minor changes.

3.1 The algebra of ribbon Schur operators

Let K denote the field $\mathbb{Q}(q)$. Let $\mathbf{F}$ denote a vector space over K spanned by a countable basis $\{\lambda \mid \lambda \in \mathcal{P}\}$ indexed by partitions. We shall call $\mathbf{F}$ the *Fock space*. Define linear operators $u_i^{(n)} : \mathbf{F} \to \mathbf{F}$ for $i \in \mathbb{Z}$ which we call *ribbon Schur operators* by:

$$u_i^{(n)} : \lambda \longmapsto \begin{cases} q^{\mathrm{spin}(\mu/\lambda)}\mu & \text{if } \mu/\lambda \text{ is a } n\text{-ribbon with head lying on the } i\text{-th diagonal,} \\ 0 & \text{otherwise.} \end{cases}$$

We will usually suppress the integer n in the notation, even though u_i depends on n. If we need to emphasize this dependence, we write $u_i^{(n)}$. We say that a partition λ has an *i-addable ribbon* if a n-ribbon can be added to λ with head on the i-th diagonal. Similarly, λ has an *i-removable ribbon* if a n-ribbon can be removed from λ with head on the i-th diagonal. Suppose the core $\tilde{\lambda}$ of λ has offset sequence $(d_0, d_1, \ldots, d_{n-1})$. Then the operator $u^{(n)}_{(d_j+k)n+j}$ acts on the j-th partition $\lambda^{(j)}$ of the n-quotient by adding a square on the k-th diagonal, and multiplying by a suitable power of q.

Observe that a skew shape λ/μ is a horizontal ribbon strip if there exists $i_1 < i_2 < \cdots < i_k$ so that $u_{i_k} u_{i_{k-1}} \cdots u_{i_2} u_{i_1} \cdot \mu = q^a \lambda$ for some integer $a = \mathrm{spin}(\lambda/\mu)$.

Let $\mathcal{U} = \mathcal{U}_n \subset \mathrm{End}_K(\mathbf{F})$ denote the algebra generated by the operators $\{u_i^{(n)}\}$ over K.

Proposition 3.1. *The operators u_i satisfy the following commutation relations:*

$$u_i u_j = u_j u_i \qquad \text{for } |i-j| \geq n+1, \tag{3.1}$$

$$u_i^2 = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{3.2}$$

$$u_{i+n} u_i u_{i+n} = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{3.3}$$

$$u_i u_{i+n} u_i = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{3.4}$$

$$u_i u_j = q^2 u_j u_i \qquad \text{for } n > i-j > 0. \tag{3.5}$$

Furthermore, these relations generate all the relations that the u_i satisfy. The subalgebra generated by $\{u_i\}_{i=1}^{i=kn}$ has dimension $(C_k)^n$ where $C_k = \frac{1}{2k+1}\binom{2k}{k}$ is the k-th Catalan number.

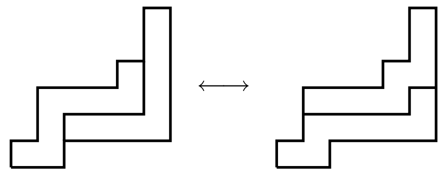

Figure 3-1: Calculating relation (3.5) of Proposition 3.1.

Proof. Relations (3.1-3.4) follow from the description of ribbon tableaux in terms of the n-quotient and the usual relations for the operators $u_i^{(1)}$ (see for example [12]). Relation (3.5) is a quick calculation (see Figure 3-1), and also follows from the inversion statistics of [51, 17] which give the spin in terms of the n-quotient.

Now we show that these are the only relations. The usual Young tableau case with $n = 1$ was shown by Billey, Jockush and Stanley in [2]. However, when $q = 1$, we are reduced to a direct product of n copies of this action as described earlier. The operators $\{u_i \mid i = k \mod n\}$ act on the k-th tableau of the n-quotient independently.

Let $f = a_u\underline{u} + a_v\underline{v} + \cdots \in \mathcal{U}$ and suppose f acts identically as 0 on $\mathbf{F}$. First suppose that some monomial $\underline{u}$ acts identically as 0. Then by the earlier remarks, the subword of $\underline{u}$ consisting only of $\{u_i \mid i = k \mod n\}$ must act identically as 0 for some k. Using relation (3.5), we see that we can deduce $\underline{u} = 0$, using the result of [2].

Now suppose that a monomial $\underline{v}$ does not act identically as 0, so that $\underline{v} \cdot \mu = q^t\lambda$ for some $t \in \mathbb{Z}$ and $\mu, \lambda \in \mathcal{P}$. Collect all other monomials $\underline{v}'$ such that $\underline{v}' \cdot \mu = q^{b(\underline{v}')+t}\lambda$ for some $b(\underline{v}') \in \mathbb{Z}$. By Lemma 3.2 below, $\underline{v}' = q^{b(\underline{v}')}\underline{v}$. Since $f \cdot \mu = 0$ we must have $\sum_{\underline{v}'} a_{v'}\underline{v}' = 0$ and by Lemma 3.2 this can be deduced from the relations (3.1-3.5). This shows that we can deduce $f = 0$ from the relations.

For the last statement of the theorem, relation (3.5) reduces the statement to the case $n = 1$. When $n = 1$, we think of the u_i as the Coxeter generators s_i of S_{k+1}. A basis of the algebra generated by $\{u_i^{(1)}\}_{i=1}^k$ is given by picking a reduced decomposition for each 321-avoiding permutation – these are exactly the permutations with no occurrences of $s_i s_{i+1} s_i$ in any reduced decomposition. It is well known that the number of these permutations is equal to a Catalan number. ∎

Lemma 3.2. *Suppose* $\underline{u} = u_{i_k}u_{i_{k-1}}\cdots u_{i_1}$ *and* $\underline{v} = u_{j_l}u_{j_{l-1}}\cdots u_{j_1}$. *If* $\underline{u}\cdot\mu = q^t\underline{v}\cdot\mu \neq 0$ *for some* $t \in \mathbb{Z}$ *and* $\mu \in \mathcal{P}$ *then* $\underline{u} = q^t\underline{v}$ *as operators on* $\mathbf{F}$, *and this can be deduced from the relations of Proposition 3.1.*

Proof. Using relation (3.5) and the n-quotient bijection, we can reduce the claim to the case $n = 1$ which we now assume. So suppose $\underline{u}\cdot\mu = \underline{v}\cdot\mu = \lambda$. Then the multiset of indices $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_l\}$ are identical, since these are the diagonals of λ/μ. In particular we have $k = l$.

We need only show that using relations (3.1-3.4) we can reduce to the case where $u_{i_1} = u_{j_1}$, and the result will follow by induction on k. Let $a = \min\{b \mid j_b = i\}$ where we set $i = i_1$. We can move u_{j_a} to the right of $\underline{v}$ unless for some $c < a$, we have $j_c = i \pm 1$. But μ has an i-addable corner, and so $\nu = u_{j_{c-1}}u_{j_{c-2}}\cdots u_{j_1}\cdot\mu$ also has an i-addable corner. This implies that $u_{i\pm1}\cdot\nu = 0$ so no such c can exist. ∎

3.2 Homogeneous symmetric functions in ribbon Schur operators

Let $h_k(\mathbf{u}) := \sum_{i_1<i_2<\cdots<i_k} u_{i_k}\cdots u_{i_1}$ be the "homogeneous" symmetric functions in the operators u_i (the name makes sense since $u_i^2=0$). Since the u_i do not commute, the ordering of the variables is important in the definition. The action of $h_k(\mathbf{u})$ on $\mathbf{F}$ is well defined even though $h_k(\mathbf{u})$ does not lie in $\mathcal{U}$ but in some completion. Alternatively, we may write

$$\prod_{i=\infty}^{i=-\infty}(1+xu_i)=\sum_{i=0}^{\infty}x^k h_k(\mathbf{u})$$

where if $i<j$ then $(1+xu_i)$ appears to the right of $(1+xu_j)$ in the product. By the remarks in Section 3.1, the operator $h_k(\mathbf{u})$ adds a horizontal ribbon strip of size k to a partition, so that

$$h_k(\mathbf{u})\cdot\lambda=\sum_{\mu}q^{\mathrm{spin}(\mu/\lambda)}\mu$$

where the sum is over all horizontal ribbon strips μ/λ of size k.

The following proposition was shown in [38] using representation theoretic results in [28] (see Chapter 4). Our new proof imitates [12, 11].

Theorem 3.3. *The elements $\{h_k(\mathbf{u})\}_{k=1}^{\infty}$ commute and generate an algebra isomorphic to the algebra of symmetric functions.*

Proof. Given any fixed partition only finitely many u_i do not annihilate it. Since we are adding only a finite number of ribbons, it suffices to prove that $h_k(u_a,u_{a+1},\ldots,u_b)$ commute for every two integers $b>a$. First suppose that $n\geq b-a$. We may assume without loss of generality that $a=1$ and $b=n+1$. We expand both $h_k(u_a,u_{a+1},\ldots,u_b)h_l(u_a,u_{a+1},\ldots,u_b)$ and $h_l(u_a,u_{a+1},\ldots,u_b)h_k(u_a,u_{a+1},\ldots,u_b)$ and collect monomials with the same set of indices $I=\{i_1<i_2<\cdots<i_{k+l}\}$. By Proposition 3.1, any operator u_i can occur at most once in any such monomial. Suppose the collection of indices I does not contain both 1 and $n+1$. Then by Proposition 3.1 we may reorder any such monomial $\underline{u}=u_{j_1}u_{j_2}\cdots u_{j_{k+l}}$. into the form $q^t u_{i_1}u_{i_2}\cdots u_{i_{k+l}}=q^t u_I$. The integer t is given by twice the number of inversions in the word $j_1j_2\cdots j_{k+l}$. That the coefficient of u_I is the same in $h_k(u_a,u_{a+1},\ldots,u_b)h_l(u_a,u_{a+1},\ldots,u_b)$ and $h_l(u_a,u_{a+1},\ldots,u_b)h_k(u_a,u_{a+1},\ldots,u_b)$ is equivalent to the following generating function identity for permutations (alternatively, symmetry of a Gaussian polynomial).

Let $D_{m,k}$ be the set of permutations of S_m with a single ascent at the k-th position and let $d_{m,k}(q)=\sum_{w\in D_{m,k}}q^{\mathrm{inv}(w)}$ where $\mathrm{inv}(w)$ denotes the number of inversions in w. Then we need the identity

$$d_{k+l,k}(q)=d_{k+l,l}(q).$$

This identity follows immediately from the involution on S_m given by $w=w_1\cdots w_m\mapsto v=v_1\cdots v_m$ where $v_i=m+1-w_{m+1-i}$.

When I contains both 1 and $n+1$ then we must split further into cases depending on the locations of these two indices: (a) $u_{n+1}\cdots u_1\cdots$; (b) $\cdots u_1u_{n+1}\cdots$; (c) $\cdots u_{n+1}\cdots u_1$; (d) $u_{n+1}\cdots u_1$. We pair case (a) of $h_k(u_a,u_{a+1},\ldots,u_b)h_l(u_a,u_{a+1},\ldots,u_b)$ with case (c) of $h_l(u_a,u_{a+1},\ldots,u_b)h_k(u_a,u_{a+1},\ldots,u_b)$ and vice versa; and also cases (b) and (d) with itself. After this pairing, and using relation (3.5) of Proposition 3.1, the argument goes as before. For example, in cases (a) and (c), we move u_{n+1} to the front and u_1 to the end.

Now we consider $h_k(u_a, \ldots, u_b)$ for $b - a > n$. Let $E_{b,a}(x) = (1+xu_b)(1+xu_{b-1})\cdots(1+xu_a)$. Note that $E_{b,a}(x)^{-1} = (1-xu_a)(1-xu_{a+1})\cdots(1-xu_b)$ is a valid element of $\mathcal{U}[x]$. The commuting of the $h_k(u_a, u_{a+1}, \ldots, u_b)$ is equivalent to the following identity:

$$E_{b,a}(x)E_{b,a}(y) = E_{b,a}(y)E_{b,a}(x)$$

as power series in x and y with coefficients in $\mathcal{U}$ which we assume to be known for all (b, a) satisfying $b - a < l$ for some $l > n$. Now let $b = a + l$. In the following we use the fact that u_a and u_b commute.

$$\begin{aligned}
&E_{b,a}(x)E_{b,a}(y) \\
&= E_{b,a+1}(x)(1+xu_a)(1+yu_b)E_{b-1,a}(y) \\
&= E_{b,a+1}(y)E_{b,a+1}(x)\left(E_{b,a+1}(y)\right)^{-1}(1+yu_b)(1+xu_a)\left(E_{b-1,a}(x)\right)^{-1}E_{b-1,a}(y)E_{b-1,a}(x) \\
&= E_{b,a+1}(y)E_{b,a+1}(x)\left(E_{b-1,a+1}(x)E_{b-1,a+1}(y)\right)^{-1}E_{b-1,a}(y)E_{b-1,a}(x) \\
&= E_{b,a+1}(y)(1+xu_b)(1+yu_a)E_{b-1,a}(x) \\
&= E_{b,a}(y)E_{b,a}(x).
\end{aligned}$$

This proves the inductive step and thus also that the $h_k(\mathbf{u})$ commute. To see that they are algebraically independent, we may restrict our attention to an infinite subset of the operators $\{u_i\}$ which all mutually commute, in which case the $h_k(\mathbf{u})$ are exactly the classical elementary symmetric functions in those variables. □

Theorem 3.3 allows us to make the following definition, following [12].

Definition 3.4. The *non-commutative (skew) Schur functions* $s_{\lambda/\mu}(\mathbf{u})$ are given by the Jacobi-Trudi formula:

$$s_{\lambda/\mu}(\mathbf{u}) = \det\left(h_{\lambda_i - i + j - \lambda_j}(\mathbf{u})\right)_{i,j=1}^{l(\lambda)}.$$

Similarly, using Theorem 3.3 one may define the non-commutative symmetric function $f(\mathbf{u})$ for any symmetric function f.

It is not clear at the moment how to write $s_\lambda(\mathbf{u})$, or even just $e_k(\mathbf{u}) = s_{1^k}(\mathbf{u})$, in terms of monomials in the u_i like in the definition of $h_k(\mathbf{u})$. One can check, for example, that $p_2(\mathbf{u})$ cannot be written as a non-negative sum of monomials.

3.3 The Cauchy identity for ribbon Schur operators

The vector space $\mathbf{F}$ comes with a natural inner product $\langle .,. \rangle$ such that $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$. Let d_i denote the adjoint operators to the u_i with respect to this inner product. They are given by

$$d_i : \lambda \longmapsto \begin{cases} q^{\mathrm{spin}(\lambda/\mu)}\mu & \text{if } \lambda/\mu \text{ is a } n\text{-ribbon with head lying on the } i\text{-th diagonal,} \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma is a straightforward computation.

Lemma 3.5. *Let $i \neq j$ be integers. Then $u_i d_j = d_j u_i$.*

Define $U(x)$ and $D(x)$ by

$$U(x) = \cdots (1 + xu_2)(1 + xu_1)(1 + xu_0)(1 + xu_{-1}) \cdots$$

and

$$D(x) = \cdots (1 + xd_{-2})(1 + xd_{-1})(1 + xd_0)(1 + xd_1) \cdots .$$

So $U(x) = \sum_k x^k h_k(\mathbf{u})$ and we similarly define $h_k^\perp(\mathbf{u})$ by $D(x) = \sum_k x^k h_k^\perp(\mathbf{u})$. The operator $h_k^\perp(\mathbf{u})$ acts by removing a horizontal ribbon strip of length k from a partition.

The main result of this section is the following identity.

Theorem 3.6. *The following "Cauchy Identity" holds:*

$$U(x)D(y) \prod_{i=0}^{n-1} \frac{1}{1 - q^{2i}xy} = D(y)U(x). \tag{3.6}$$

A combinatorial proof of this identity was given by Marc van Leeuwen [42] via an explicit shape datum for a Schensted-correspondence. Our proof is suggested by ideas in [11] but considerably different to the techniques there. The following Corollary is immediate after equating coefficients of $x^a y^b$ in Theorem 3.6.

Corollary 3.7. *Let $a, b \geq 1$ and $m = min(a, b)$. Then*

$$h_b^\perp(\mathbf{u})h_a(\mathbf{u}) = \sum_{i=0}^{m} h_i(1, q^2, \ldots, q^{2(n-1)})h_{a-i}(\mathbf{u})h_{b-i}^\perp(\mathbf{u}).$$

Define the operators $h_i^{(j)} = (u_i d_i)^j - (d_i u_i)^j$ for $i, j \in \mathbb{Z}$ and $j \geq 1$. The operators $h_i^{(j)}$ act as follows:

$$h_i^{(j)} : \lambda \mapsto \begin{cases} -q^{2j\cdot\mathrm{spin}(\mu/\lambda)}\lambda & \text{if } \lambda \text{ has an } i\text{-addable ribbon } \mu/\lambda, \\ q^{2j\cdot\mathrm{spin}(\lambda/\nu)}\lambda & \text{if } \lambda \text{ has an } i\text{-removable ribbon } \lambda/\nu, \\ 0 & \text{otherwise.} \end{cases}$$

Since $h_i^{(j)}$ acts diagonally on $\mathbf{F}$ in the natural basis, they all commute with each other.

We will need the following proposition, due to van Leeuwen [42].

Proposition 3.8. *Let λ be a partition and suppose ribbons R_i and R_j can be added or removed on diagonals $i < j$ such that no ribbons can be added or removed on a diagonal $d \in (i, j)$. Then one of the following holds:*

1. *Both R_i and R_j can be added and* $\mathrm{spin}(R_j) = \mathrm{spin}(R_i) - 1$.
2. *Both R_i and R_j can be removed and* $\mathrm{spin}(R_j) = \mathrm{spin}(R_i) + 1$.
3. *One of R_i and R_j can be added and the other can be removed and* $\mathrm{spin}(R_j) = \mathrm{spin}(R_i)$.

For example, writing down, from bottom left to top right, the spins of the ribbons that can be added and removed from the partition in Figure 1-1 gives $2, 1, -1, 1, -1, 1, 0$ where positive numbers denote addable ribbons and negative numbers denote removable ribbons.

Lemma 3.9. *Let λ be a partition, $i \in \mathbb{Z}$ and $j \geq 1$ be an integer. Suppose that λ has an i-addable ribbon with spin s. Then*

$$\left(\sum_{k=i+1}^{k=\infty} h_k^{(j)}\right) \lambda = -(1 + q^{2j} + \cdots + q^{2(s-1)j})\lambda.$$

Suppose that a λ has an i-removable ribbon with spin s.. Then

$$\left(\sum_{k=i+1}^{k=\infty} h_k^{(j)}\right) \lambda = -(1 + q^{2j} + \cdots + q^{2sj})\lambda.$$

Also, if i is sufficiently small so that no ribbons can be added to λ before the i-th diagonal, then

$$\left(\sum_{k=i}^{k=\infty} h_k^{(j)}\right) \lambda = -(1 + q^{2j} + \cdots + q^{2(n-1)j})\lambda.$$

Proof. This follows from Proposition 3.8 and the fact that the furthest ribbon to the right (respectively, left) that can be added to any partition always has spin 0 (respectively, $n-1$). □

Lemma 3.10. *Let $i, j \in \mathbb{Z}$ and $j \geq 1$. Then*

$$u_i \left(\sum_{k=i+1}^{k=\infty} h_k^{(j)}\right) = \left(\sum_{k=i}^{k=\infty} h_k^{(j)}\right) u_i.$$

Similarly we have,

$$d_i \left(\sum_{k=i}^{k=\infty} h_k^{(j)}\right) = \left(\sum_{k=i+1}^{k=\infty} h_k^{(j)}\right) d_i.$$

Proof. We consider applying the first statement to a partition λ. The expression vanishes unless λ has an i-addable ribbon μ/λ, which we assume has spin s. Then we can compute both sides using Lemma 3.9. The second statement follows similarly. □

We may rewrite equation (3.6) as

$$U(x)D(y) = D(y) \prod_{i=0}^{n-1} (1 - q^i xy) U(x).$$

Now

$$\prod_{i=0}^{n-1} (1 - q^{2i} xy) = \sum_{i=0}^{n-1} (-1)^i e_i(1, q^2, \ldots, q^{2(n-1)})(xy)^i$$

and $\sum_{i=-\infty}^{i=\infty} h_i^{(j)}$ acts as the scalar $-p_j(1, q^2, \ldots, q^{2(n-1)}) = -(1 + q^{2j} + \cdots + q^{2(n-1)j})$ by Lemma 3.9. Let us use the notation $p_j(h_i) = -\sum_{k=i}^{\infty} h_k^{(j)}$, for $i \in \mathbb{Z} \cup \{\infty\}$. We also write

$$e_n(h_i) = \sum_{\rho \vdash n} \epsilon_\rho z_\rho^{-1} p_\rho(h_i)$$

where $p_\rho(h_i) = p_{\rho_1}(h_i)p_{\rho_2}(h_i)\cdots$ and $\epsilon_\rho = (-1)^{|\rho|-l(\rho)}$ and z_ρ is as defined earlier. As scalar operators on $\mathbf{F}$ we have,

$$\prod_{j=0}^{n-1}(1-q^{2j}xy) = \sum_{j=0}^{n}(-1)^j e_j(h_{-\infty})(xy)^j = \sum_{j=0}^{\infty}(-1)^j e_j(h_{-\infty})(xy)^j.$$

Lemma 3.11. *Let $i \in \mathbb{Z}$. Then*

$$(1+yd_i)\left(\sum_{k=0}^{\infty}(-1)^k e_k(h_i)(xy)^k\right)(1+xu_i) = (1+xu_i)\left(\sum_{k=0}^{\infty}(-1)^k e_k(h_{i+1})(xy^k)\right)(1+yd_i).$$

Proof. First we consider the coefficient of $x^{k+1}y^k$. We need to show that $e_k(h_i)u_i = u_i e_k(h_{i+1})$ which just follows immediately from the definition of $e_k(h_i)$ and Lemma 3.10. The equality for the coefficient of $x^k y^{k+1}$ follows in a similar manner.

Now consider the coefficient of $(xy)^k$. We need to show that

$$d_i e_{k-1}(h_i)u_i - e_k(h_i) - u_i e_{k-1}(h_{i+1})d_i + e_k(h_{i+1}) = 0.$$

By Lemma 3.10, this is equivalent to

$$((u_i d_i - h_i)e_{k-1}(h_{i+1}) - e_k(h_i) - u_i d_i e_{k-1}(h_i) + e_k(h_{i+1}))\cdot\lambda = 0 \tag{3.7}$$

for every partition λ. We now split into three cases depending on λ.

1. Suppose that that λ has a i-addable ribbon with spin s. Then $d_i\cdot\lambda = 0$ so (3.7) reduces to $\big(e_k(h_{i+1}) + q^{2s}e_{k-1}(h_{i+1}) - e_k(h_i)\big)\cdot\lambda = 0$. Using Lemma 3.9, this becomes

$$e_k(1,q^2,\ldots,q^{2(s-1)}) + q^{2s}e_{k-1}(1,q^2,\ldots,q^{2(s-1)}) = e_k(1,q^2,\ldots,q^{2s})$$

   which is an easy symmetric function identity.

2. Suppose that that λ has a i-removable ribbon with spin s. Then $u_i d_i\cdot\lambda = h_i\cdot\lambda$ so we are reduced to showing

$$e_k(h_{i+1}) - e_k(h_i) - q^{2s}e_{k-1}(h_i) = 0$$

   which by Lemma 3.9 becomes the same symmetric function identity as above.

3. Suppose that λ has neither a i-addable or i-removable ribbon. Then (3.7) becomes $e_k(h_i)\cdot\lambda = e_k(h_{i+1})\cdot\lambda$ so the result is immediate.

□

Theorem 3.6 now follows easily.

Proof of Theorem 3.6. We need to show that $U(x)D(y)\cdot\lambda = D(y)\prod_{i=0}^{n-1}(1-q^{2i}xy)U(x)\cdot\lambda$ for each partition λ. But if we focus on a fixed coefficient $x^a y^b$, then for some integers $s<t$ depending on λ, a and b, we may assume that $u_i = d_i = 0$ for $i<s$ and $i>t$ for all our computations. Thus it suffices to show that

$$(1+xu_s)\cdots(1+xu_t)(1+yd_t)\cdots(1+yd_s) =$$
$$(1+yd_t)\cdots(1+yd_s)\left(\textstyle\sum_{k=0}^{\infty}(-1)^k e_k(h_s)(xy)^k\right)(1+xu_s)\cdots(1+xu_t).$$

Since by Lemma 3.5, $(1+xu_a)$ and $(1+yd_b)$ commute unless $a = b$, this follows by applying Lemma 3.11 repeatedly. □

3.4 Ribbon functions and q-Littlewood Richardson Coefficients

In this section we connect the non-commutative Schur functions $s_\nu(\mathbf{u})$ with the q-Littlewood Richardson coefficients $c^\nu_{\lambda/\mu}(q)$. The set up here is actually a special case of work of Fomin and Greene [12], though not all of their assumptions and results apply in our context.

Let $x_1, x_2, \ldots$ be commutative variables. Consider the Cauchy product in the commutative variables $\{x_i\}$ and non-commutative variables $\{u_i\}$ (not to be confused with the Cauchy identity in Section 3.3):

$$\Omega(x, \mathbf{u}) = \prod_{j=1}^{\infty} \left(\prod_{i=\infty}^{\infty} (1 + x_j u_i) \right)$$

where the product is multiplied so that terms with smaller j are to the right and for the same j, terms with smaller i are to the right. We have

$$\Omega(x, \mathbf{u}) = \prod_{j=1}^{\infty} \left(\sum_k x_j^k h_k(\mathbf{u}) \right) = \sum_\lambda s_\lambda(X) s_\lambda(\mathbf{u}) = \sum_\lambda m_\lambda(x) h_\lambda(\mathbf{u})$$

where we have used Theorem 3.3 and the classical Cauchy identity for symmetric functions.

Since each $h_k(\mathbf{u})$ adds a horizontal ribbon strip, we see that

$$\mathcal{G}_{\lambda/\mu}(X; q) = \sum_\alpha x^\alpha \langle h_\alpha(\mathbf{u}) \cdot \mu, \lambda \rangle = \sum_\nu m_\nu(x) \langle h_\nu(\mathbf{u}) \cdot \mu, \lambda \rangle = \langle \Omega(x, \mathbf{u}) \cdot \mu, \lambda \rangle$$

where the sum is over all compositions α or all partitions ν. In the language of Fomin and Greene [12], these functions were denoted $F_{g/h}$. The symmetry of the ribbon functions follows from the fact that $h_\alpha(\mathbf{u}) = h_\beta(\mathbf{u})$ if the compositions α and β are rearrangements of each other.

Theorem 3.12. *The power series $\mathcal{G}_{\lambda/\mu}(X; q)$ are symmetric functions in the (commuting) variables $\{x_1, x_2, \ldots\}$ with coefficients in K.*

Other elementary proofs of Theorem 3.12 appear in [16, 34].

Let $\langle ., . \rangle_X$ be the (Hall) inner product in the X variables. The action of the noncommutative Schur functions in ribbon Schur operators $s_\nu(\mathbf{u})$ calculate the skew q-Littlewood Richardson coefficients.

Lemma 3.13. *The q-Littlewood coefficients are given by $c^\nu_{\lambda/\mu}(q) = \langle s_\nu(\mathbf{u}) \cdot \mu, \lambda \rangle$.*

Proof. We can write

$$\begin{aligned} c^\nu_{\lambda/\mu}(q) &= \left\langle \mathcal{G}_{\lambda/\mu}(X; q), s_\nu(X) \right\rangle_X \\ &= \langle \langle \Omega(x, \mathbf{u}) \cdot \mu, \lambda \rangle, s_\nu(X) \rangle_X \\ &= \sum_\rho \langle s_\rho(X) \langle s_\rho(\mathbf{u}) \cdot \mu, \lambda \rangle, s_\nu(X) \rangle_X \\ &= \langle s_\nu(\mathbf{u}) \cdot \mu, \lambda \rangle \end{aligned}$$

□

Proposition 3.14. *The noncommutative Schur function $s_\nu(\mathbf{u})$ can be written as a non-negative sum of monomials if and only if the skew q-Littlewood Richardson coefficients $c^\nu_{\lambda/\mu}(q) \in \mathbb{N}[q]$ are non-negative polynomials for all skew shapes λ/μ.*

Proof. The only if direction is trivial since $\langle \underline{u} \cdot \mu, \lambda \rangle$ is always a non-negative polynomial in q for any monomial $\underline{u}$ in the u_i. Suppose $c^\nu_{\lambda/\mu}(q)$ are non-negative polynomials for all skew shapes λ/μ. Write $s_\nu(\mathbf{u})$ as an alternating sum of monomials. Let $\underline{u}$ and $\underline{v}$ be two monomials occurring in $s_\nu(\mathbf{u})$. If $\underline{u} \cdot \mu = \underline{v} \cdot \mu \neq 0$ for some partition μ then by Lemma 3.2, $\underline{u} = \underline{v}$. Now collect all terms $\underline{v}$ in $s_\nu(\mathbf{u})$ such that $\underline{u} \cdot \mu = q^{a(\underline{v})} \underline{v} \cdot \mu$. Collecting all monomials $\underline{v}$ with a fixed $a(\underline{v})$, we see that we must be able to cancel out any negative terms since all coefficients in $c^\nu_{\lambda/\mu}(q) = \langle s_\nu(\mathbf{u}) \cdot \mu, \lambda \rangle$ are non-negative. □

To our knowledge, a combinatorial proof of the non-negativity of the skew q-Littlewood Richardson coefficients is only known for the case $n = 2$ via the Yamanouchi domino tableaux of Carré and Leclerc [5]. When $\mu = \emptyset$, Leclerc and Thibon [40] have shown using results of [60], that the q-Littlewood Richardson coefficients $c^\nu_\lambda(q)$ are non-negative polynomials in q; see also Section 4.1.3.

3.5 Non-commutative Schur functions in ribbon Schur operators

This section is logically independent of the remainder of the thesis. Based on Proposition 3.14, we suggest the following problem, which is the main problem of this Chapter.

Conjecture 3.15. *We have:*

1. *The non-commutative Schur functions $s_\lambda(\mathbf{u})$ can be written as a non-negative sum of monomials.*

2. *A canonical such expression for $s_\lambda(\mathbf{u})$ can be given by picking some monomials occurring in $h_\lambda(\mathbf{u})$. That is, $s_\lambda(\mathbf{u})$ is a positive sum of monomials $\underline{u} = u_{i_k} u_{i_{k-1}} \cdots u_{i_1}$ where $i_{\lambda_1} > i_{\lambda_1 - 1} > \cdots > i_1$ and $i_{\lambda_1+\lambda_2} > i_{\lambda_1+\lambda_2-1} > \cdots > i_{\lambda_1+1}$ and so on, where $k = |\lambda|$.*

By the usual Littlewood-Richardson rule, Conjecture 3.15 also implies that the skew Schur functions $s_{\lambda/\mu}(\mathbf{u})$ can be written as a non-negative sum of monomials. If Conjecture 3.15 is true, we propose the following definition.

Conjectural Definition 3.16. *Let λ/μ be a skew shape and ν a partition so that $n|\nu| = |\lambda/\mu|$. Let $\underline{u} = u_{i_k} u_{i_{k-1}} \cdots u_{i_1}$ be a monomial occurring in (a canonical expression of) $s_\nu(\mathbf{u})$ so that $\underline{u}.\mu = q^a \lambda$ for some integer a. If Conjecture 3.15.(2) holds, then the action of $\underline{u}$ on μ naturally corresponds to a ribbon tableau of shape λ/μ and weight ν. We call such a tableau a* Yamanouchi ribbon tableau.

In their work, Fomin and Greene [12] show that if the u_i satisfy certain relations then $s_\lambda(\mathbf{u})$ can be written in terms of the reading words of semistandard tableaux of shape λ. These relations do not hold for our ribbon Schur operators. We shall see that a similar description holds for our $s_\lambda(\mathbf{u})$ when λ is a hook shape, but appears to fail for other shapes.

The following theorem holds for any variables u_i satisfying Theorem 3.3. The commutation relations of Proposition 3.1 are not needed at all. Let T be a tableau (not necessarily semistandard). The *reading word* reading(T) is obtained by reading beginning in the top row from right to left and then going downwards. If $w = w_1 w_2 \cdots w_k$ is a word, then we set $u_w = u_{w_1} u_{w_2} \cdots u_{w_k}$.

Theorem 3.17. *Let $\lambda = (a, 1^b)$ be a hook shape. Then*

$$s_\lambda(\mathbf{u}) = \sum_T u_{\text{reading}(T)} \tag{3.8}$$

where the summation is over all semistandard tableaux T of shape λ. For our purposes, the semistandard tableaux can be filled with any integers not just positive ones, and the row and columns both satisfy strict inequalities (otherwise $u_{\text{reading}(T)} = 0$).

Proof. The theorem is true by definition when $b = 0$. We proceed by induction on b, supposing that the theorem holds for partitions of the form $(a, 1^{b-1})$. Let $\lambda = (a, 1^b)$. The Jacobi-Trudi formula gives

$$s_\lambda(\mathbf{u}) = \det \begin{pmatrix} h_a(\mathbf{u}) & h_{a+1}(\mathbf{u}) & \cdots & h_{a+b-1}(\mathbf{u}) & h_{a+b}(\mathbf{u}) \\ 1 & h_1(\mathbf{u}) & \cdots & h_{b-1}(\mathbf{u}) & h_b(\mathbf{u}) \\ 0 & 1 & \cdots & h_{b-2}(\mathbf{u}) & h_{b-1}(\mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & h_1(\mathbf{u}) \end{pmatrix}.$$

Expanding the determinant beginning from the bottom row we obtain

$$s_\lambda(\mathbf{u}) = \sum_{j=1}^{j=b} \left((-1)^{j+1} s_{(a,1^{b-j})}(\mathbf{u}) h_j(\mathbf{u}) \right) + (-1)^b h_{a+b}(\mathbf{u}).$$

Using the inductive hypothesis, $s_{(a,1^{b-j})}(\mathbf{u}) h_j(\mathbf{u})$ is the sum over all the monomials $\underline{u} = u_{i_1} u_{i_2} \cdots u_{i_{a+b}}$ satisfying $i_1 > i_2 > \cdots > i_a < i_{a+1} < \cdots < i_{a+b-j}$ and $i_{a+b-j+1} > i_{a+b-j+2} > \cdots > i_{a+b}$. Let A_j be the sum of those monomials also satisfying $i_{a+b-j} < i_{a+b-j+1}$ and B_j be the sum of those such that $i_{a+b-j} > i_{a+b-j+1}$ so that $s_{(a,1^{b-j})}(\mathbf{u}) h_j(\mathbf{u}) = A_j + B_j$. Observe that $B_j = A_{j+1}$ for $j \neq b$ and that $B_b = h_{a+b}(\mathbf{u})$. Cancelling these terms we obtain $s_\lambda(\mathbf{u}) = A_1$ which completes the inductive step and the proof. ☐

In particular, when λ is a column we obtain the following theorem.

Theorem 3.18. *Let $k \geq 1$. The operator $e_k(\mathbf{u})$ acts on $\mathbf{F}$ by adding vertical ribbon strips, so that*

$$e_k(\mathbf{u}) \cdot \lambda = \sum_\mu q^{\text{spin}(\mu/\lambda)} \mu$$

where the sum is over all vertical ribbon strips μ/λ of size k and spin(μ/λ) *denotes the spin of the tiling of μ/λ as a vertical ribbon strip.*

We speculate that when $n > 2$, the formula (3.8) of Theorem 3.17 holds if and only if λ is a hook shape. Combining Theorem 3.17 with the results of [16] one can obtain Macdonald positivity for coefficients of Schur functions labelled by hook shapes.

2	3
0	1

0	2
1	3

1	2
0	4

0	1
2	3

0	2
1	4

1	2
0	3

Figure 3-2: The bottom row contains some 3-commuting tableaux and the top row contains some tableaux which are not 3-commuting.

We now describe $s_\lambda(\mathbf{u})$ for shapes λ of the form $(s, 2)$. We will only need the following definition for these shapes, but we make the general definition in the hope it may be useful for other shapes.

Definition 3.19. Let $T : \{(x, y) \in \lambda\} \to \mathbb{Z}$ be a filling of the squares of λ, where x is the row index, y is the column index and the numbering for x and y begins at 1. Then T is a *n-commuting* tableau if the following conditions hold:

1. All rows are increasing, that is, $T(x, y) < T(x, y + 1)$ for $(x, y), (x, y + 1) \in \lambda$.

2. If $(x, y), (x + 1, y) \in \lambda$ and $y > 1$ then $T(x, y) \leq T(x + 1, y)$. Also if $y = 1$ and $(x + 1, 2) \notin \lambda$ then $T(x, 1) \leq T(x + 1, 1)$.

3. Suppose the two-by-two square $(x, 1), (x, 2), (x + 1, 1), (x + 1, 2)$ lies in λ for some y. Then if $T(x, 1) > T(x + 1, 1)$ we must have $T(x + 1, 2) - T(x + 1, 1) \leq n$. Otherwise $T(x, 1) < T(x + 1, 1)$ and either we have $T(x, 2) \leq T(x + 1, 1)$ or we have both $T(x, 2) > T(x + 1, 1)$ and $T(x + 1, 2) - T(x, 1) \leq n$.

We give some examples of commuting and non-commuting tableaux of shape $(2, 2)$ in Figure 3-2.

Let us now note that the operators u_i satisfy the following Knuth-like relations, using Proposition 3.1. Let i, j, k satisfy either $i < j < k$ or $i > j > k$, then depending on whether u_i and u_k commute, we have

1. Either $u_i u_k u_j = u_k u_i u_j$ or $u_i u_k u_j = u_j u_i u_k$.

2. Either $u_j u_k u_i = u_j u_i u_k$ or $u_j u_k u_i = u_k u_i u_j$.

Note that the statement of these relations do not explicitly depend on n.

Theorem 3.20. *Let $\lambda = (s, 2)$. Then*

$$s_\lambda(\mathbf{u}) = \sum_T u_{\text{reading}}(T)$$

where the summation is over all n-commuting tableaux T of shape λ.

Proof. By definition $s_\lambda(\mathbf{u}) = h_s(\mathbf{u})h_2(\mathbf{u}) - h_{s+1}(\mathbf{u})h_1(\mathbf{u})$. We have that $h_{s+1}(\mathbf{u})h_1(\mathbf{u})$ is equal to the sum over monomials $\underline{u} = u_{i_1} \cdots u_{i_{s+1}} u_{i_{s+2}}$ such that $i_1 > i_2 > \cdots > i_{s+1}$. We will show how to use the Knuth-like relations (1) and (2) to transform each such monomial into one that occurs in $h_s(\mathbf{u})h_2(\mathbf{u})$, in an injective fashion.

Let $a = i_{s-1}$, $b = i_s$, $c = i_{s+1}$ and $d = i_{s+2}$ so that $a > b > c$. We may assume that a, b, c, d are all different for otherwise $\underline{u} = 0$ by Prop 3.1. If $c > d$ then $\underline{u}$ is already a monomial occurring in $h_s(\mathbf{u})h_2(\mathbf{u})$. So suppose $c < d$. Now if $d > b > c$ we apply the Knuth-like relations to transform $\underline{v} = u_a u_b u_c u_d$ to either $\underline{v}' = u_a u_b u_d u_c$ or $\underline{v}' = u_a u_c u_d u_b$, where we always pick the former if u_d and u_c commute. If $b > d > c$ we may transform $\underline{v} = u_a u_b u_c u_d$ to either $\underline{v}' = u_a u_c u_b u_d$ or $\underline{v}' = u_a u_d u_b u_c$ again picking the former if u_b and u_c commute. In all cases the resulting monomial occurs in $h_s(\mathbf{u})h_2(\mathbf{u})$ and the map $\underline{v} \mapsto \underline{v}'$ is injective if the information of whether u_c commutes with both u_b and u_d is fixed. Writing $\{j_0 < j_1 < j_2 < j_3\}$ for $\{a, b, c, d\}$ to indicate the relative order we may tabulate the possible resulting monomials $\underline{v}'$ as reading words of the following "tableaux":

$$a > b > c > d: \quad \begin{matrix} j_2 & j_3 \\ j_0 & j_1 \end{matrix} \quad (j_0 < j_1 < j_2 < j_3 \in \mathbb{Z});$$

$$a > b > d > c: \quad \begin{matrix} j_0 & j_3 \\ j_1 & j_2 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_2} \text{ commute})$$

$$\begin{matrix} j_1 & j_3 \\ j_0 & j_2 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_2} \text{ do not commute});$$

$$a > d > b > c: \quad \begin{matrix} j_1 & j_3 \\ j_0 & j_2 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_2} \text{ commute})$$

$$\begin{matrix} j_0 & j_3 \\ j_1 & j_2 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_2} \text{ do not commute});$$

$$d > a > b > c: \quad \begin{matrix} j_1 & j_2 \\ j_0 & j_3 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_3} \text{ commute})$$

$$\begin{matrix} j_0 & j_2 \\ j_1 & j_3 \end{matrix} \quad (u_{j_0} \text{ and } u_{j_3} \text{ do not commute}).$$

Finally, cancelling these monomials from $h_s(\mathbf{u})h_2(\mathbf{u})$, we see that the monomials in $s_\lambda(\mathbf{u})$ are of the form $u_{i_1} \cdots u_{i_{s-3}} u_{i_{s-2}} u_{\mathrm{reading}(T)}$ for a tableau T of the following form, (satisfying $i_{s-2} > t$ where t is the value in the top right hand corner):

$$\begin{matrix} j_0 & j_1 \\ j_2 & j_3 \end{matrix} \quad \text{with no extra conditions,}$$

$$\begin{matrix} j_1 & j_2 \\ j_0 & j_3 \end{matrix} \quad \text{if } u_{j_3} \text{ and } u_{j_0} \text{ do not commute,}$$

$$\begin{matrix} j_0 & j_2 \\ j_1 & j_3 \end{matrix} \quad \text{if } u_{j_3} \text{ and } u_{j_0} \text{ commute,}$$

where in the first case $j_0 < j_1 \leq j_2 < j_3$ and in the remaining cases $j_0 < j_1 < j_2 < j_3$. We may allow more equalities to be weak, but the additional monomials $u_{\mathrm{reading}(T)}$ that we obtain turn out all to be 0. One immediately checks that these monomials are the reading words of n-commuting tableaux of shape $(s, 2)$. $\square$

By Theorem 3.17, $e_k(\mathbf{u}) = \sum_{i_1 < i_2 < \cdots < i_k} u_{i_1} \cdots u_{i_k}$. Thus using the dual Jacobi-Trudi

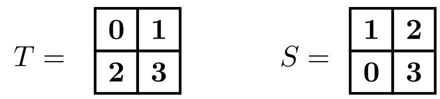

Figure 3-3: The 3-commuting tableaux with shape $(2,2)$ and squares filled with $\{0,1,2,3\}$.

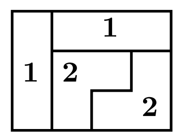

Figure 3-4: The Yamanouchi ribbon tableau corresponding to $c^{(2,2)}_{(4,4,4)}(q) = q^4$.

formula, we can get a description for $s_{\lambda'}(\mathbf{u})$ whenever we have one for $s_\lambda(\mathbf{u})$ by reversing the order '$<$' on $\mathbb{Z}$. This for example leads to a combinatorial interpretation for $s_\lambda(\mathbf{u})$ of the form $\lambda = (2,2,1^a)$ which we will not write out explicitly.

One can also obtain different combinatorial interpretations for $s_\lambda(\mathbf{u})$ by for example changing $h_a(\mathbf{u})h_2(\mathbf{u})$ to $h_2(\mathbf{u})h_a(\mathbf{u})$ in the proof of Theorem 3.20. This leads to the reversed reading order on tableaux which also has the order '$<$' reversed. In the case $n = 2$, the n-commuting tableaux are nearly the same as usual semistandard tableaux where row and column inequalities are strict. The reversed reading order on order-reversed semistandard tableaux would lead to the same combinatorial interpretation as Carré and Leclerc's Yamanouchi domino tableaux [5].

It seems likely that Theorem 3.17 and Theorem 3.20 may be combined to give a description of $s_\lambda(\mathbf{u})$ for $\lambda = (a, 2, 1^b)$ but so far we have been unable to make progress on the case $\lambda = (3,3)$.

We end this section with an example.

Example 3.21. Let $n = 3$ and $\lambda = (4,4,4)$. Let us calculate the coefficient of s_{22} in $\mathcal{G}_\lambda(X;q)$. The shape λ has ribbons on the diagonals $\{0,1,2,3\}$, so we are concerned with 3-commuting tableaux of shape $(2,2)$ filled with the numbers $\{0,1,2,3\}$. There are only two such tableaux S and T given in Figure 3-3. It is easy to see that $u_{\text{reading}(T)}.\emptyset = 0$ so the coefficient of s_{22} in $\mathcal{G}_\lambda(X;q)$ is given by the spin of the ribbon tableau corresponding to $u_{\text{reading}(S)}.\emptyset = u_2u_1u_3u_0.\emptyset$. This tableau has spin 4, so that $c^{(2,2)}_{(4,4,4)}(q) = q^4$ (see Figure 3-4).

We may check directly that in fact

$$\mathcal{G}_\lambda(X;q) = q^2 s_{211} + q^4(s_{31} + s_{22}) + q^6 s_{31} + q^8 s_4.$$

3.6 Final remarks on ribbon Schur operators

3.6.1 Maps involving $\mathcal{U}_n$

The algebra $\mathcal{U}_n$ has many automorphisms. The map sending $u_i \mapsto u_{i+1}$ is an algebra isomorphism of $\mathcal{U}_n$ and the map sending $u_i \mapsto u_{-i}$ a semi-linear algebra involution.

Proposition 3.22. *There are commuting injections*

$$\mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow \mathcal{U}_3 \cdots$$

and surjections

$$\cdots \mathcal{U}_3 \twoheadrightarrow \mathcal{U}_2 \twoheadrightarrow \mathcal{U}_1.$$

Proof. The injection $\mathcal{U}_{n-1} \hookrightarrow \mathcal{U}_n$ can be given by sending $u_{k(n-1)+i}$ to u_{kn+i} where $i \in \{0, 1, \ldots, n-2\}$, though there are other choices. The surjection $\mathcal{U}_n \twoheadrightarrow \mathcal{U}_{n-1}$ is given by sending u_{kn+i} to $u_{k(n-1)+i}$ for $i \in \{0, 1, \ldots, n-2\}$ and sending u_{kn+n-1} to 0 for all $k \in \mathbb{Z}$. $\square$

3.6.2 The algebra $\mathcal{U}_\infty$

Picking compatible injections as above, the inductive limit $\mathcal{U}_\infty$ of the algebras $\mathcal{U}_n$ has a countable set of generators $u_{i,j}$ (the image of $u^{(n)}_{i+jn}$) where $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. The generators are partially ordered by the relation $(i, j) < (k, l)$ if either $j < l$, or $j = l$ and $i < k$. The generators satisfy the following relations:

$$\begin{aligned} u_{i,j}u_{k,l} &= u_{k,l}u_{i,j} && \text{if } |j-l| \geq 2 \text{ or if } j = l \pm 1 \text{ and } i \neq k, \\ u_{i,j}^2 &= 0 && \text{for any } i, j \in \mathbb{Z}, \\ u_{i,j}u_{i,j\pm 1}u_{i,j} &= 0 && \text{for any } i, j \in \mathbb{Z}, \\ u_{i,j}u_{k,j} &= q^2 u_{k,j}u_{i,j} && \text{for } i, j, k \in \mathbb{Z} \text{ satisfying } i > k, \\ u_{i,j+1}u_{k,j} &= q^2 u_{k,j}u_{i,j+1} && \text{for } i, j, k \in \mathbb{Z} \text{ satisfying } i < k. \end{aligned}$$

Many of the results of this paper can be phrased in terms of $\mathcal{U}_\infty$. For example, one can define ∞-commuting tableaux which are maps $T : \lambda \to \mathbb{N} \times \mathbb{Z}$.

3.6.3 Another description of $\mathcal{U}_n$

There is an alternative way of looking at the algebras $\mathcal{U}_n \subset \mathrm{End}(\mathbf{F})$. Let $u = u_0$ be the operator adding a n-ribbon on the 0-th diagonal. As explained in Section 2.1 we may view a partition λ in terms of its $\{0, 1\}$-*edge sequence* $\{p_i(\lambda)\}_{i=-\infty}^{\infty}$. Let t be the operator which shifts a bit sequence $\{p_i\}_{i=-\infty}^{\infty}$ one-step to the right, so that $(t \cdot p)_i = p_{i-1}$. Note that t does not send a partition to a partition so we need to consider it as a linear operator on a larger space (spanned by doubly-infinite bit sequences, for example). It is clear that $u_i = t^i u t^{-i}$ and we may consider $\mathcal{U}_n$ as sitting inside an enlarged algebra $\tilde{\mathcal{U}_n} = K[u, t, t^{-1}]$.

3.6.4 Connection with work of van Leeuwen and Fomin

Van Leeuwen [42] has given a spin-preserving Robinson-Schensted-Knuth correspondence for ribbon tableaux. The calculations in Section 3.3 are essentially an algebraic version of van Leeuwen's correspondence, in a manner similar to the construction of the generalised Schensted correspondences in [11]. It would be interesting to make these relationships precise. This should, for example, yield purely bijective proofs of the ribbon Pieri rule (Theorem 4.12) and the symmetry of ribbon functions.

Chapter 4

Ribbon Pieri and Cauchy Formulae

In this chapter we will use the results of Chapter 3 concerning the operators $h_k(\mathbf{u})$ and $h_k^{\perp}(\mathbf{u})$ to deduce properties of the ribbon functions $\mathcal{G}_{\lambda/\mu}(X;q)$. The operators $\{u_i \mid i \in \mathbb{Z}\}$ will no longer be used in our computations. The results in this chapter will appear in [33] in a different form.

Let K denote the field $\mathbb{Q}(q)$ as before. Define a representation $\phi : \Lambda(q) \to \mathrm{End}_K(\mathbf{F})$ of the symmetric functions on the Fock space by

$$\phi : h_k \longmapsto h_k(\mathbf{u}).$$

By Proposition 3.3 and the fact that $\{h_k\}$ generate $\Lambda(q)$, this definition extends to a representation of $\Lambda(q)$. Now let $\psi : \Lambda \to \mathrm{End}_K(\mathbf{F})$ be the adjoint representation of $\Lambda(q)$ on $\mathbf{F}$, with respect to the inner product $\langle .,. \rangle$. It is defined by $\psi(h_k) = h_k^{\perp}(\mathbf{u})$.

For convenience we shall define $S_\lambda := \phi(s_\lambda) = s_\lambda(\mathbf{u})$.

4.1 Ribbon functions and the Fock space

4.1.1 The action of the Heisenberg algebra on F

The *Heisenberg Algebra* H is the associative algebra with 1 generated over K by a countable set of generators $\{B_k : k \in \mathbb{Z}\setminus\{0\}\}$ satisfying

$$[B_k, B_l] = l \cdot a_l(q) \cdot \delta_{k,-l} \tag{4.1}$$

for some elements $a_l(q) \in K$ satisfying $a_l(q) = a_{-l}(q)$. (Often the element 1 is called the central element and denoted c, but we will not need this generality). The *Bosonic Fock space representation* $K[H_-]$ of H is the polynomial algebra

$$K[H_-] := K[B_{-1}, B_{-2}, \ldots].$$

The elements B_{-k} for $k \geq 1$ act by multiplication on $K[H_-]$. The action of B_k for $k \geq 1$ is given by (4.1) and the relation $B_k \cdot 1 = 0$ for $k \geq 1$.

An explicit construction of $K[H_-]$ is given by $\Lambda(q)$. We may identify B_k as the following operators:

$$B_k \longmapsto \begin{cases} f \longmapsto a_{-k}(q)p_{-k} \cdot f & \text{for } k < 0 \\ f \longmapsto k\frac{\partial}{\partial p_k} f & \text{for } k > 0. \end{cases}$$

Under this identification, the operators B_k have degree $-k$.

A standard lemma that we shall need later is

Lemma 4.1. *Let $k \geq 1$ be an integer and λ be a partition. Then*

$$B_k B_{-\lambda} = k a_k(q) m_k(\lambda) B_{-\mu} + B_{-\lambda} B_k$$

where $m_k(\lambda)$ is the number of parts of λ equal to k and μ is λ with one less part equal to k. If $m_k(\lambda) = 0$ to begin with then the first term is just 0.

Proof. We may commute B_k with $B_{-\lambda_i}$ immediately for parts $\lambda_i \neq k$. For each part equal to k, using the relation $[B_{-k}, B_k] = k a_k(q)$ introduces one term of the form $k a_k(q) B_{-\mu}$. □

Combining Theorem 3.6 with Theorem 3.3 we obtain the following Theorem, first proved in [28] (the connection with ribbon tableaux was first shown in [38]):

Theorem 4.2. *The maps ϕ and ψ generate and action of the Heisenberg algebra H with parameters $a_l(q) = 1 + q^{2l} + \cdots + q^{2(n-1)l}$ for $l > 0$ on $\mathbf{F}$. The representation $\Theta : H \to \mathrm{End}_K(\mathbf{F})$ is given by*

$$\vartheta : B_k \longmapsto \begin{cases} \phi(p_{-k}) & \text{if } k < 0 \\ \psi(p_k) & \text{if } k > 0. \end{cases}$$

Thus we have

$$[\phi(p_k), \psi(p_l)] = k \frac{1 - q^{2nk}}{1 - q^{2k}} \delta_{k,l}. \tag{4.2}$$

Proof. When k and l have the same sign, the commutation relation follows from Theorem 3.3. For the other case, we first write (see [46, 55])

$$h_a(\mathbf{u}) = \sum_{\lambda \vdash a} z_\lambda^{-1} p_\lambda(\mathbf{u}),$$

where $z_\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots m_1(\lambda)! m_2(\lambda)! \cdots$ and $m_i(\lambda)$ denotes the number of parts of λ equal to i. We first show that (4.2) implies Corollary 3.7. Thus we need to show that (4.2) implies

$$\left(\sum_{\lambda \vdash b} z_\lambda^{-1} p_\lambda^\perp(\mathbf{u}) \right) \left(\sum_{\lambda \vdash a} z_\lambda^{-1} p_\lambda(\mathbf{u}) \right)$$
$$= \sum_{i=0}^{m} h_i(1, q^2 \ldots, q^{2(n-1)}) \left(\sum_{\lambda \vdash a-i} z_\lambda^{-1} p_\lambda(\mathbf{u}) \right) \left(\sum_{\lambda \vdash b-i} z_\lambda^{-1} p_\lambda^\perp(\mathbf{u}) \right).$$

Note that $p_k(1, q, \ldots, q^{2(n-1)}) = \frac{1-q^{2n|k|}}{1-q^{2|k|}}$. Let μ and ν be partitions such that $m = |\nu| = |\mu|$. One checks that the coefficient of $p_\mu(\mathbf{u}) p_\nu^\perp(\mathbf{u})$ on the right hand side is equal to $z_\nu^{-1} z_\mu^{-1} \sum_{\lambda \vdash m} z_\lambda^{-1} p_\lambda(1, q^2, \ldots, q^{2(n-1)})$. Let $\rho = \lambda \cup \mu$ and $\pi = \lambda \cup \nu$. We claim that the summand $z_\nu^{-1} z_\mu^{-1} z_\lambda^{-1} p_\lambda(1, q^2, \ldots, q^{2(n-1)})$ is the coefficient of $p_\mu(\mathbf{u}) p_\nu^\perp(\mathbf{u})$ when applying (4.2) repeatedly to $z_\pi^{-1} z_\rho^{-1} p_\pi^\perp(\mathbf{u}) p_\rho(\mathbf{u})$. This is a straightforward computation, counting the number of ways of picking parts from ρ and π to make the partition λ.

Thus (4.2) implies Corollary 3.7, and since both the homogeneous and power sum symmetric functions generate the algebra of symmetric functions, Corollary 3.7 must be equivalent to (4.2).

□

Note that this action of the Heisenberg algebra differs from the one in the literature by the change of variables $q \mapsto -q^{-1}$. The operators $B_{-k} = \phi(p_k)$ and $B_k = \psi(p_k)$ are known as *Bosonic operators*. For the remainder of this chapter, the Heisenberg algebra will always refer to the algebra with parameters $a_l(q) = 1 + q^{2l} + \cdots + q^{2(n-1)l}$ for $l > 0$.

For later use, we also define $\mathcal{X}^k_{\lambda/\mu}(q) \in \mathbb{Q}[q]$ by $B_{-k} \cdot \mu = \sum_\lambda \mathcal{X}^k_{\lambda/\mu}(q) \cdot \lambda$ for $k > 0$. Since B_k is adjoint to B_{-k} with respect to $\langle .,. \rangle$, we also have $B_k \cdot \lambda = \sum_\lambda \mathcal{X}^k_{\lambda/\mu}(q) \cdot \mu$ for $k > 0$. We will show in Section 4.4.3 that the coefficients $\mathcal{X}^k_{\lambda/\mu}(q)$ can be described in terms of "border ribbon strips".

4.1.2 The action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$

In this section we introduce the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ and describe its action on the Fock Space $\mathbf{F}$. The connection with $U_q(\widehat{\mathfrak{sl}}_n)$ was the original motivation for the study of the action of H on $\mathbf{F}$. We will not use the details of this description but we include the details for completeness. A concise introduction to the material of this section can be found in [39]. Throughout q can be thought of as either a formal parameter of as a generic complex number (not equal to a root of unity). We also assume familiarity with root systems.

We denote by $\mathfrak{h}$ the Cartan subalgebra of $\widehat{\mathfrak{s}l}_n$ which is spanned over $\mathbb{C}$ by the basis $\{h_0, h_1, \ldots, h_{n-1}, D\}$. The dual basis is denoted by $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}, \delta\}$. We set $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$ for $i \in \{1, 2, \ldots, n-2\}$, and $\alpha_0 = \Lambda_0 - \Lambda_{n-1} - \Lambda_1 + \delta$ and $\alpha_{n-1} = -\Lambda_{n-2} + \Lambda_{n-1} - \Lambda_0$. The generalised Cartan matrix $[\langle \alpha_i, h_j \rangle]$ will be denoted a_{ij}. Set $P^\vee = (\oplus_{i=0}^{n-1} \mathbb{Z} h_i) \oplus \mathbb{Z} D$.

The algebra $U_q(\widehat{\mathfrak{sl}}_n)$ is the associative algebra over K generated by elements e_i, f_i for $0 \le i \le n-1$, and q^h for $h \in P^\vee$ satisfying the following relations:

$$q^h q^{h'} = q^{h+h'},$$

$$q^{h_i} e_j q^{-h_i} = q^{a_{ij}} e_j, \quad q^{h_i} f_j q^{-h_i} = q^{-a_{ij}} f_j,$$

$$q^D e_i q^{-D} = \delta_{i0} q^{-1} e_0, \quad q^D f_i q^{-D} = \delta_{i0} q f_0,$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j).$$

We have used the standard notation

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k][k-1] \cdots [1],$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

The slightly smaller quantized enveloping algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ is defined by the same generators and relations as $U_q(\widehat{\mathfrak{sl}}_n)$ except that the Cartan part is replaced by q^h for $h \in P^\vee/\mathbb{Z}D$. In other words $U'_q(\widehat{\mathfrak{sl}}_n)$ is the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ missing the generator q^D.

There is an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}$ due to Hayashi [18] which was formulated essentially as follows by Misra and Miwa [48].

Recall that a cell (i,j) has content given by $c(i,j) = i - j$. Its residue $p(i,j) \in \{0, 1, \ldots, n-1\}$ is then $i - j \mod n$. We call (i,j) an indent k-node of λ if $p(i,j) = k$ and $\lambda \cup (i,j)$ is a valid Young diagram. We make the analogous definition for a removable i-node.

Let $i \in \{0, 1, \ldots, n-1\}$ and $\mu = \lambda \cup \delta$ for an indent i-node δ of λ. Now set

$N_i(\lambda) = \#\{\text{indent } i\text{-nodes of } \lambda\} - \#\{\text{removable } i\text{-nodes of } \lambda\}$.

$N_i^l(\lambda, \mu) = \#\{\text{indent } i\text{-nodes of } \lambda \text{ to the left of } \delta \text{ (not counting } \delta)\} - \#\{\text{removable } i\text{-nodes of } \lambda \text{ to the left of } \delta\}$.

$N_i^r(\lambda, \mu) = \#\{\text{indent } i\text{-nodes of } \lambda \text{ to the right of } \delta \text{ (not counting } \delta)\} - \#\{\text{removable } i\text{-nodes of } \lambda \text{ to the left of } \delta\}$.

$N^0(\lambda) = \#\{0 \text{ nodes of } \lambda\}$.

Then we have the following theorem.

Theorem 4.3. *The following formulae define an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}$:*

$$\begin{aligned}
q^{h_i} \cdot \lambda &= q^{N_i(\lambda)} \lambda, \text{ for each } i \in \{0, 1, \ldots, n-1\}, \\
q^D \cdot \lambda &= q^{N^0(\lambda)} \lambda, \\
f_i \cdot \lambda &= \sum_\mu q^{N_i^r(\lambda,\mu)} \mu, \text{ summed over all } \mu \text{ such that } \mu/\lambda \text{ has residue } i, \\
e_i \cdot \lambda &= \sum_\mu q^{-N_i^l(\lambda,\mu)} \mu, \text{ summed over all } \mu \text{ such that } \lambda/\mu \text{ has residue } i.
\end{aligned}$$

The $U'_q(\widehat{\mathfrak{sl}}_n)$-submodule of $\mathbf{F}$ generated by the vector $\emptyset$ can be seen to be the irreducible highest weight module with highest weight Λ_0, which we will denote V_{Λ_0}.

Kashiwara, Miwa and Stern [28] have defined an action of the the affine Hecke algebra $\hat{H}_N$ on the tensor product $V(z)^{\otimes N}$ of evaluation modules for the vector representation V of $U'_q(\widehat{\mathfrak{sl}}_n)$. As $N \to \infty$, it is shown in [28] that one obtains an action of the center $Z(\hat{H}_N)$ as a copy of the Heisenberg algebra H on $\mathbf{F}$, commuting with the action of $U'_q(\widehat{\mathfrak{sl}}_n)$.

Theorem 4.4 ([28]). *The representation Θ of H commutes with the action of the quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$. The Fock space $\mathbf{F}$, regarded as a representation of $U'_q(\widehat{\mathfrak{sl}}_n) \otimes H$ decomposes as the tensor product*

$$\mathbf{F} \simeq V_{\Lambda_0} \otimes K[H_-]$$

where $K[H_-]$ is the Fock space of the Heisenberg algebra H and V_{Λ_0} is the highest weight representation with highest weight Λ_0.

We will return to quantum affine algebras in Chapter 5.

4.1.3 Relation to upper global bases of F

Recall that Kashiwara [25] and Lusztig [45] have shown that irreducible representations of quantum groups possess distinguished bases known as *global crystal bases*. The Fock space $\mathbf{F}$ is not an irreducible representation of $U_q(\widehat{\mathfrak{sl}}_n)$ but nevertheless, Lascoux, Leclerc and Thibon [37, 40, 41] have defined and studied two global bases in $\mathbf{F}$, which agree with the global crystal bases when we restrict to an irreducible component of $\mathbf{F}$ under the action of $U_q(\widehat{\mathfrak{sl}}_n)$. We will only be interested in the *upper global basis*. Note that throughout this section, our notation differs from that of the literature by the change of variables $q \mapsto -q^{-1}$.

The map $v \mapsto \overline{v}$ of the following Proposition is known as the *bar involution* and was defined by Leclerc and Thibon [40, 41].

Proposition 4.5. *There exists a unique semi-linear map* $^- : \mathbf{F} \to \mathbf{F}$ *satisfying for each* $v \in \mathbf{F}$,

$$\begin{aligned}
\overline{qv} &= q^{-1}\overline{v}, \\
\overline{f_i \cdot v} &= f_i \cdot \overline{v}, \\
\overline{e_i \cdot v} &= e_i \cdot \overline{v}, \\
\overline{B_{-k} \cdot v} &= B_{-k} \cdot \overline{v}, \\
\overline{B_k \cdot v} &= -q^{-2(n-1)k} B_k \cdot \overline{v}.
\end{aligned}$$

This involution restricted to V_{Λ_0} (the $U_q(\widehat{\mathfrak{sl}}_n)$ submodule with highest weight vector indexed by the empty partition) agrees with Kashiwara's involution [25].

Theorem 4.6 ([40]). *There exist unique vectors* $G_\lambda \in \mathbf{F}$ *for* $\lambda \in \mathcal{P}$ *satisfying:*

$$\overline{G_\lambda} = G_\lambda \quad \text{and} \quad G_\lambda \equiv \lambda \text{ mod } q\mathcal{L}^-$$

where $\mathcal{L}^-$ *is the* $\mathbb{Z}[q]$ *submodule of* $\mathbf{F}$ *spanned by* $\{\lambda \mid \lambda \in \mathcal{P}\}$.

When we restrict this to $V_{\Lambda_0} \subset \mathbf{F}$, the G_λ is essentially the global upper crystal basis of V_{Λ_0} (see [37, 26]). The following result connects the upper global basis with ribbon functions.

Theorem 4.7 ([40]). *The upper global basis element* $G_{n\lambda}$ *is given by*

$$G_{n\lambda} = S_\lambda \cdot \emptyset.$$

It follows immediately that $\{G_{n\lambda} \mid \lambda \in \mathcal{P}\}$ form a basis of the space of highest weight vectors of the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}$. By Lemma 3.13, we have $G_{n\lambda} = \sum_\mu c^\lambda_\mu(q)\mu$ and so the (non-skew) q-Littlewood Richardson coefficients $c^\lambda_\mu(q)$ are certain coefficients of the upper global basis.

Remark 4.8. Let

$$G_\lambda = \sum_\mu l_{\lambda,\mu}(q)\mu,$$

for some polynomials $l_{\lambda,\mu}(q) \in \mathbb{Q}[q]$. Varagnolo and Vasserot [60] have shown that $l_{\lambda,\mu}(q)$ is a parabolic Kazhdan-Lusztig polynomial for the affine Hecke algebra of type A. These polynomials were introduced by Deodhar [6] and were shown to have non-negative coefficients by Kashiwara and Tanisaki [29], using the geometry of affine Schubert varieties. This implies that the q-Littlewood Richardson coefficients $c^{\lambda}_{\mu}(q) \in \mathbb{N}[q]$ are polynomials with non-negative coefficients; see [40].

4.2 The map $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$

Define a representation $\Theta^* : H \to \mathrm{End}_K(\Lambda(q))$ by

$$B_k \longmapsto \begin{cases} k\frac{\partial}{\partial p_k} & \text{for } k > 0 \\ \left(\frac{1-q^{2nk}}{1-q^{2k}}\right) p_k & \text{for } k < 0. \end{cases}$$

Definition 4.9. Let $\mathbf{\Phi} : \mathbf{F} \to \Lambda(q)$ be the $\mathbb{Q}(q)$-linear map defined by

$$\lambda \longmapsto \mathcal{G}_{\lambda/\tilde{\lambda}}(X; q).$$

The map $\mathbf{\Phi}$ has remarkable properties. It converts linear properties in $\mathbf{F}$ into algebraic properties in $\Lambda(q)$. In [38], the Fock space $\mathbf{F}$ was identified with Λ and $\mathbf{\Phi}$ called "the adjoint of the q-plethysm operator", though its properties were not studied there.

The following theorem shows that $\mathbf{\Phi}$ can be thought of as a projection of $\mathbf{F}$ onto the Fock space $K[H_-]$ of H. It is a generalisation of the Boson-Fermion correspondence; see Chapter 5.

Theorem 4.10. *The map $\mathbf{\Phi}$ is a map of H-modules. More precisely,*

$$\mathbf{\Phi} \circ \Theta = \Theta^* \circ \mathbf{\Phi}$$

as maps from H to $\mathrm{Hom}_K(\mathbf{F}, \Lambda(q))$.

Proof. We will check that $\mathbf{\Phi} \circ \Theta(B_k) = \Theta^*(B_k) \circ \mathbf{\Phi}$ for each k. Abusing notation, we do not distinguish B_k and $\Theta(B_k)$ from now on.

Let δ be an n-core which we fix throughout and suppose $k \geq 1$. We will calculate the expression $B_k S_\lambda \cdot \delta$ in two ways. By Lemma 3.13 we can write

$$B_k S_\lambda \cdot \delta = \sum_{\mu \in \mathcal{P}_\delta} c^{\lambda}_{\mu/\delta}(q) B_k \cdot \mu = \sum_{\nu \in \mathcal{P}_\delta} \left(\sum_{\mu \in \mathcal{P}_\delta} c^{\lambda}_{\mu/\delta}(q) \mathcal{X}^k_{\mu/\nu}(q) \right) \nu.$$

On the other hand, we can compute $B_k S_\lambda$ within H. We note that $B_k \cdot \delta = 0$, which follows from the definition $B_k = p_k^{\perp}(\mathbf{u})$ and the fact we cannot remove any ribbons from the shape δ. Thus by Lemma 4.1, we have

$$B_k S_\lambda \cdot \delta = \left(\frac{1-q^{2nk}}{1-q^{2k}}\right) \sum_{\mu} \chi^k_{\lambda/\mu} S_\mu \cdot \delta$$

where the coefficients $\chi^k_{\lambda/\mu}$ are given by $p_k^{\perp} s_\lambda = \sum_\mu \chi^k_{\lambda/\mu} s_\mu$ in Λ. By Lemma 3.13 again, we find that this is equal to

$$\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_{\mu}\chi^k_{\lambda/\mu}\sum_{\nu\in\mathcal{P}_\delta}c^{\mu}_{\nu/\delta}(q)\cdot\nu=\sum_{\nu\in\mathcal{P}_\delta}\left(\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_{\mu}\chi^k_{\lambda/\mu}c^{\mu}_{\nu/\delta}(q)\right)\nu.$$

Equating coefficients of $\cdot\nu$ we obtain

$$\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_{\mu}\chi^k_{\lambda/\mu}c^{\mu}_{\nu/\delta}(q)=\sum_{\mu\in\mathcal{P}_\delta}c^{\lambda}_{\mu/\delta}(q)\mathcal{X}^k_{\mu/\nu}(q). \tag{4.3}$$

We now calculate

$$\begin{aligned}\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)p_k\mathcal{G}_{\nu/\delta}(X;q)&=\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_{\mu\in\mathcal{P}_\delta}c^{\mu}_{\nu/\delta}(q)p_ks_\mu\\&=\left(\frac{1-q^{2nk}}{1-q^{2k}}\right)\sum_{\mu}c^{\mu}_{\nu/\delta}(q)\left(\sum_{\lambda}\chi^k_{\lambda/\mu}s_\lambda\right)\\&=\sum_{\lambda}\left(\sum_{\mu\in\mathcal{P}_\delta}c^{\lambda}_{\mu/\delta}(q)\mathcal{X}^k_{\mu/\nu}(q)\right)s_\lambda\quad\text{using Equation (4.3)}\\&=\sum_{\mu\in\mathcal{P}_\delta}\mathcal{X}^k_{\mu/\nu}(q)\mathcal{G}_{\mu/\delta}(X;q),\end{aligned}$$

which is equivalent to $\Theta^*(B_{-k})\cdot\mathbf{\Phi}(\nu)=\mathbf{\Phi}(\Theta(B_{-k})\cdot\nu)$. This is true for all ν and proves the claim for $k<0$. The other case follows similarly.

□

We have proved Theorem 4.10 by a calculation expressing ribbon functions in the Schur basis. A similar calculation using other bases is certainly possible; see Section 5.2.2. Theorem 4.10 gives the following Corollary.

Corollary 4.11. *Let λ be a partition and δ a n-core. The following identity holds in $\Lambda(q)$.*

$$s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]=\sum_{\mu\in\mathcal{P}_\delta}c^{\lambda}_{\mu/\delta}(q)\mathcal{G}_{\mu/\delta}(X;q)=\sum_{\mu\in\mathcal{P}_\delta,\ \nu\in\mathcal{P}}c^{\lambda}_{\mu/\delta}(q)c^{\nu}_{\mu/\delta}(q)s_\nu(X).$$

Proof. These are immediate consequences of Theorem 4.10 and Lemma 3.13 as $\mathbf{\Phi}(\delta)=1$ for an n-core δ. □

Note that by Theorem 4.7, we can identify the image of the global basis element G_{n_λ} under $\mathbf{\Phi}$:

$$\mathbf{\Phi}(G_{n\lambda})=s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X].$$

4.3 Ribbon Pieri formulae

Define the formal power series

$$H(t)=\prod_{i\geq1}\prod_{k=0}^{n-1}\frac{1}{1-x_iq^{2k}t}\ ;\ E(t)=\prod_{i\geq1}\prod_{k=0}^{n-1}\left(1+x_iq^{2k}t\right).$$

These power series are completely natural in the context of Robinson-Schensted ribbon insertion where they are spin-weight generating functions for sets of ribbons; see Section 4.9. Suppressing the notation for n, we define symmetric functions $\mathbf{h}_k$ and $\mathbf{e}_k$ by

$$H(t) = \sum_k \mathbf{h}_k t^k \ ; \ E(t) = \sum_k \mathbf{e}_k t^k.$$

In plethystic notation, $\mathbf{h}_k = h_k[(1+q^2+\cdots+q^{2(n-1)})X]$ and $\mathbf{e}_k = e_k[(1+q^2+\cdots+q^{2(n-1)})X]$. The following theorem is an immediate consequence Theorem 4.10, the definition of the plethysm $h_k[(1+q^2+\cdots+q^{2(n-1)})X]$ and Theorem 3.18.

Theorem 4.12 (Ribbon Pieri Rule). *Let λ be a partition with n-core δ. Then*

$$\mathbf{h}_k \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{\mathrm{spin}(\mu/\lambda)} \mathcal{G}_{\mu/\delta}(X;q) \tag{4.4}$$

where the sum is over all partitions μ such that μ/λ is a horizontal n-ribbon strip with k ribbons. Also

$$\mathbf{e}_k \mathcal{G}_{\lambda/\delta}(X;q) = \sum_\mu q^{\mathrm{spin}(\mu/\lambda)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all partitions μ such that μ/λ is a vertical n-ribbon strip with k ribbons.

When $n = 1$, we recover the classical Pieri rule for Schur functions. We can obtain the two statements of Theorem 4.12 from each other via the involution ω_n of Section 4.5. Let $\mathrm{mspin}(\lambda)$ denote the maximum spin of a ribbon tableau of shape λ. By Theorem 4.12, we have

$$\mathbf{h}_k = \sum_\lambda q^{\mathrm{mspin}(\lambda)} \mathcal{G}_\lambda(X;q) \tag{4.5}$$

where the sum is over all λ with no n-core such that $|\lambda| = kn$ with no more than n rows.

Example 4.13. Let $n = 3$, $k = 2$ and $\lambda = (3,1)$. Then

$$\mathbf{h}_2 \mathcal{G}_{(3,1)} = \mathcal{G}_{(9,1)} + q\mathcal{G}_{(6,2,2)} + q^2\mathcal{G}_{(4,4,2)} + q^2\mathcal{G}_{(6,1,1,1,1)} + q^3\mathcal{G}_{(3,3,2,1,1)} + q^4\mathcal{G}_{(3,2,2,2,1)}.$$

Setting $q = 1$ in $H(t)$ we see that $\mathbf{h}_k(X;1) = \sum_\alpha h_\alpha$ where the sum is over all compositions $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ satisfying $\alpha_0 + \cdots + \alpha_{n-1} = k$. We may thus interpret Theorem 4.12 at $q = 1$ in terms of the n-quotient as the following formula:

$$\left(\sum_\alpha h_\alpha\right) s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_\alpha \left(h_{\alpha_0} s_{\lambda^{(0)}}\right) \cdots \left(h_{\alpha_{n-1}} s_{\lambda^{(n-1)}}\right) \tag{4.6}$$

where the sum is over the same set of compositions as above. Note that the right hand side of (4.6) is indeed equal to the right hand side of (4.4) at $q = 1$ since a horizontal ribbon strip of size k is just a union of horizontal strips with total size k in the n-quotient.

We also obtain lowering versions of the Pieri rules. By Theorem 4.10 again, we have

Proposition 4.14 (Ribbon Pieri Rule – Dual Version). *Let λ be a partition with*

n-core δ and $k \geq 1$ be an integer. Then

$$h_k^{\perp} \mathcal{G}_{\lambda/\delta}(X;q) = \sum_{\mu} q^{\mathrm{spin}(\lambda/\mu)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all μ such that λ/μ is a horizontal ribbon strip. Similarly,

$$e_k^{\perp} \mathcal{G}_{\lambda/\delta}(X;q) = \sum_{\mu} q^{\mathrm{spin}(\lambda/\mu)} \mathcal{G}_{\mu/\delta}(X;q)$$

where the sum is over all μ such that λ/μ is a vertical ribbon strip.

This is a spin version of a branching formula first observed by Schilling, Shimozono and White [51] (see Section 4.7).

4.4 The ribbon Murnaghan-Nakayama rule

4.4.1 Border ribbon strip tableaux

Let

$$s_\lambda(X) = \sum_{\mu} z_\mu^{-1} \chi_\mu^\lambda p_\mu$$

be the expansion of the Schur functions in the power sum basis. When $\mu = (k)$ has only one part then we will write χ_k^λ for χ_μ^λ. The coefficients χ_μ^λ are the values of the character of $S_{|\lambda|}$ indexed by λ on the conjugacy class indexed by μ. The classical Murnaghan-Nakayama rule gives a combinatorial interpretation of these numbers:

$$\chi_\mu^\lambda = \sum_T (-1)^{h(T)}$$

where the sum is over all border-strip tableaux of shape λ and type μ. The numbers χ_μ^λ are in fact the characters of the irreducible representation labelled by λ of the symmetric group $S_{|\lambda|}$, where μ is the type of the conjugacy class. See for example [55, Ch 7.18].

More generally, we have (see [55, 46])

Proposition 4.15. *Let λ be a partition and α be a composition. Expand*

$$p_\alpha s_\lambda(X) = \sum_{\mu} \chi_\alpha^{\mu/\lambda} s_\mu(X).$$

Then $\chi_\alpha^{\mu/\lambda}$ is given by

$$\chi_\alpha^{\mu/\lambda} = \sum_T (-1)^{h(T)}$$

where the sum is over all border strip tableaux T of shape μ/λ and type α.

Note that the border strip tableaux here should not be confused with ribbon tableaux. A border strip tableau may have border strips of different sizes. A ribbon tableau has all ribbons of length n. Proposition 4.15 is usually shown algebraically using the expression of the Schur function as a bialternant $s_\lambda = a_{\lambda+\delta}/a_\delta$. Theorem 4.20 will imply that it can also be derived formally from the Pieri formula. We now generalise border strip tableaux to border ribbon strip tableaux.

Definition 4.16. A *border ribbon strip* S is a connected skew shape λ/μ with a distinguished tiling by disjoint non-empty horizontal ribbon strips $S_1, \ldots, S_a$ such that the diagram $S_{+i} = \cup_{j \leq i} S_j$ is a valid skew shape for every i and for each connected component C of S_i we have

1. The set of ribbons $C \cup S_{i-1}$ do not form a horizontal ribbon strip. Thus C has to "touch" S_{i-1} "from below".

2. No sub horizontal ribbon strip C' of C which can be added to S_{i-1} satisfies the above property. Since C is connected, this is equivalent to saying that only the rightmost ribbon of C touches S_{i-1}.

We further require that S_1 is connected. The *height* $h(S_i)$ of the horizontal ribbon strip S_i is the number of its components (two squares are connected if they share a side, but not if they only share a corner). The height $h(S)$ of the border ribbon strip is defined as $h(S) = (\sum_i h(S_i)) - 1$. The *size* of the border ribbon strip S is then the total number of ribbons in $\cup_i S_i$. A *border ribbon strip tableau* is a chain $T = \lambda_0 \subset \lambda_1 \cdots \subset \lambda_r$ of shapes together with a structure of a border ribbon strip for each skew shape λ_i/λ_{i-1}. The *type* of $T = \lambda_0 \subset \lambda_1 \cdots \subset \lambda_r$ is then the composition α with α_i equal to the size of λ_i/λ_{i-1}. The height $h(T)$ is the sum of the heights of the composite border ribbon strips. Define $\widetilde{\mathcal{X}}^{\nu}_{\mu/\lambda}(q) \in \mathbb{Z}[q]$ by

$$\widetilde{\mathcal{X}}^{\nu}_{\mu/\lambda}(q) = \sum_T (-1)^{h(T)} q^{\mathrm{spin}(T)}$$

summed over all border ribbon strip tableaux of shape μ/λ and type ν. We will show in Section 4.4.2 that $\widetilde{\mathcal{X}}^{\nu}_{\mu/\lambda}(q) = \mathcal{X}^{\nu}_{\mu/\lambda}(q)$.

Note that this definition reduces to the usual definition of a border strip and border strip tableau when $n = 1$, in which case all the horizontal strips T_i of a border ribbon strip must be connected.

Example 4.17. Let $n = 2$ and $\lambda = (4,2,2,1)$. Suppose S is a border ribbon strip such that S_1 has shape $(7,5,2,1)/(4,2,2,1)$, and thus it has size 3 and spin 1. We will now determine all the possible horizontal ribbon strips which may form S_2. It suffices to find the possible connected components that may be added. The domino $(9,5,2,1)/(7,5,2,1)$ may not be added since its union with S_1 is a horizontal ribbon strip, violating the conditions of the definition. The domino strip $(8,8,2,1)/(7,5,2,1)$ is not allowed since the domino $(8,8,2,1)/(7,7,2,1)$ can be removed and we still obtain a strip which touches S_1.

The connected horizontal ribbon strips C which can be added are $(7,7,2,1)/(7,5,2,1)$, $(7,5,3,3,2,1)/(7,5,2,1)$ and $(7,5,4,1)/(7,5,2,1)$ as shown in Figure 4-1. Thus assuming S_2 is non-empty, there are 5 choices for S_2, corresponding to taking some compatible combination of the three connected horizontal ribbon strips above.

Example 4.18. Let $n = 2$. We will calculate $\widetilde{\mathcal{X}}^{5}_{\lambda/\mu}(q)$ for $\lambda = (5,5,2)$ and $\mu = (2)$. The relevant border ribbon strips S are (successive differences of the following chains denote the S_i)

- $(2) \subset (5,5,2)$ with height 0 and spin 5,
- $(2) \subset (5,3,2) \subset (5,5,2)$ with height 1 and spin 3,

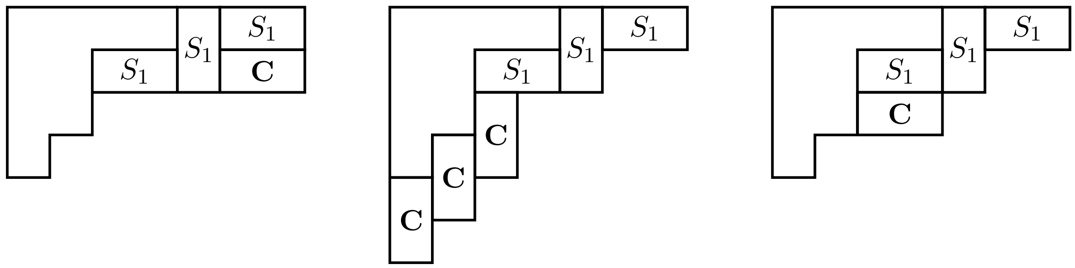

Figure 4-1: Calculation of the connected horizontal strips C which can be added to the shape $S_1 = (7,5,2,1)/(4,2,2,1)$ to form a border ribbon strip. The resulting border ribbon strips all have height 1.

- $(2) \subset (5,5) \subset (5,5,2)$ with height 1 and spin 3,
- $(2) \subset (5,3) \subset (5,5,2)$ with height 2 and spin 1.

Thus $\widetilde{\mathcal{X}}^5_{\lambda/\mu}(q) = q^5 - 2q^3 + q$.

The condition on a horizontal ribbon strip to be connected can be described in terms of the n-quotient as follows. Let T be a ribbon tableau with n-quotient $\{T^{(0)}, \ldots, T^{(n-1)}\}$. Let $\{(d_i, p_i)\}$ be the set of diagonals which are nonempty in the n-quotient of the horizontal ribbon strip R: thus diagonal diag_{d_i} of $T^{(p_i)}$ contains a square corresponding to some ribbon in the horizontal ribbon strip R. Then the horizontal ribbon strip R is connected if and only if the set of integers $\{d_i\}$ is an interval (connected) in $\mathbb{Z}$. Thus border ribbon strips may be characterised in terms of the n-quotient.

4.4.2 Formal relationship between Murnaghan-Nakayama and Pieri rules

Let V be a vector space over K and $\{v_\lambda\}_{\lambda \in \mathcal{P}}$ be a set of vectors in V labelled by partitions. Suppose $\{P_k\}$ are commuting linear operators satisfying

$$P_k v_\lambda = \sum_\mu \widetilde{\mathcal{X}}^k_{\mu/\lambda}(q) v_\mu \quad \text{for all } k, \tag{4.7}$$

then we will say that the ribbon Murnaghan-Nakayama rule holds for $\{P_k\}$. Suppose $\{H_k\}$ are commuting linear operators on V satisfying

$$H_k v_\lambda = \sum_\mu \mathcal{K}_{\mu/\lambda,k}(q) v_\mu \quad \text{for all } k, \tag{4.8}$$

then we will say that the ribbon Pieri formula holds for $\{H_k\}$.

If the skew shapes μ/λ are replaced by λ/μ in the above formulae, we get adjoint versions of these formulae which can be thought of as lowering operator formulae. Thus if a set of commuting linear operators $\{P^*_k\}$ satisfies

$$P^*_k v_\lambda = \sum_\mu \widetilde{\mathcal{X}}^k_{\lambda/\mu}(q) v_\mu \quad \text{for all } k,$$

then we will say the lowering ribbon Murnaghan-Nakayama rule holds, and similarly for the lowering ribbon Pieri rule. We begin by observing the following easy lemma.

Lemma 4.19. *The power sum and homogeneous symmetric functions satisfy:*

$$mh_m = p_{m-1}h_1 + p_{m-2}h_2 + \cdots + p_m.$$

Proof. See Chapter I, (2.10) in [46]. $\square$

Theorem 4.20. *Fix $n \geq 1$ as usual. Let $\{H_k\}$ and $\{P_k\}$ be commuting sets of linear operators, acting on a K-vector space V, satisfying the relations of Lemma 4.19 between h_k and p_k in Λ. Then the ribbon Murnaghan-Nakayama rule (4.7) holds for $\{P_k\}$ if and only if the ribbon Pieri rule (4.8) holds for $\{H_k\}$ (with respect to the same set of vectors $\{v_\lambda\}_{\lambda\in\mathcal{P}}$). An analogous statement holds for the lowering versions of the respective rules.*

Proof. Let us suppose that (4.7) holds. We will proceed by induction on k. Since $H_1 = P_1$ and a border ribbon strip of size 1 is exactly the same as a horizontal ribbon strip of size 1, the starting condition is clear. Now suppose the proposition has been shown up to $k-1$. By assumption, kH_k acts on V in the same way that $H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k$ does.

We first consider the coefficient of v_μ in $(H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k) \cdot v_\lambda$ by formally applying the rules (4.7) and (4.8). We obtain one term for each pair (S, T) where S is a border ribbon strip of size between 1 and k satisfying $\mathrm{sh}(S) = \nu/\lambda$ (for some ν) and T is a horizontal ribbon strip of size $k - \mathrm{size}(S)$ satisfying $\mathrm{sh}(T) = \mu/\nu$. Denote by $(S_1, \ldots, S_a)$ the distinguished decomposition of S into horizontal ribbon strips.

Construct a directed graph $G_{\lambda,\mu,k}$ with vertices labelled by such pairs $\mathcal{S} = \{(S,T)\}$. We have an edge

$$(S, T) \longrightarrow (S - S_a, T \cup S_a) \tag{4.9}$$

for every pair (S, T) such that $a > 1$ and $T \cup S_a$ is a horizontal strip (with the induced tiling). We claim that every non isolated connected component W of $G_{\lambda,\mu,k}$ is an inward pointing star. Indeed, every vertex must have outdegree or indegree equal to 0, and the maximum outdegree is 1, since by Condition 1 of Definition 4.16 the right hand vertex of (4.9) has outdegree 0.

Let us consider a vertex (S', T') (where $S' = \{S'_1, \ldots, S'_a\}$) with non-zero indegree. Now let C be a component of T' such that $C \cup S'_a$ is not a horizontal ribbon strip. Then there is a unique sub-horizontal ribbon strip C' of C which can be added to S' to form a border ribbon strip, by Condition 2 of Definition 4.16. This C' may be described as follows. Order the ribbons of C from left to right $c_1, c_2, \ldots, c_l$. Find the smallest i such that c_i touches the bottom of S'_a and we set $C' = \{c_1, c_2, \ldots, c_i\}$. We call such a horizontal ribbon strip C' an *addable strip* of T' (with respect to S').

A non-isolated connected component $W_{(S',T')}$ of $G_{\lambda,\mu,k}$ contains exactly of such a vertex (S', T') together with the pairs (S, T) such that $S = \{S'_1, \ldots, S'_a, S_{a+1}\}$, and S_{a+1} is the union of some (arbitrary) subset of the set of addable strips of T'. It is immediate from the construction that (S, T) will be a valid pair in $\mathcal{S}$. The contribution of $W_{(S',T')}$ to the coefficient of v_μ in $(H_{k-1}P_1 + H_{k-2}P_2 + \cdots + P_k) \cdot v_\lambda$ is

$$\sum_{(S,T)\in W_{(S',T')}} (-1)^{h(S)} q^{\mathrm{spin}(S\cup T)} = (-1)^{h(S')} q^{\mathrm{spin}(S'\cup T')} \sum_{\{C'\}} (-1)^{|\{C'\}|}$$

where on the right hand side, $\{C'\}$ varies over arbitrary subsets of addable strips of T' (we have used the fact that the tiling never changes so the spin is constant, together with the definition of height). This contribution is 0, corresponding to the identity $(1-1)^c = 0$ where c is the number of addable strips of T'.

It remains to consider the contribution of the isolated vertices: these are pairs (S, T) where $S = (S_1)$ is a connected horizontal ribbon strip such that $S \cup T$ is also a horizontal ribbon strip. Since S is connected we can recover it from $S \cup T$ by specifying its rightmost ribbon, by Condition 1 of Definition 4.16. Thus such pairs occur exactly k times for each horizontal ribbon strip of shape μ/λ, and hence the ribbon Pieri rule (4.8) is satisfied for the operator H_k.

The converse and dual claims follow from the same argument. □

4.4.3 Application to ribbon functions

It is now clear that the action of the bosonic operators B_k on $\mathbf{F}$ can be described in terms of border ribbon strips.

Theorem 4.21. *We have* $\mathcal{X}^k_{\lambda/\mu}(q) = \widetilde{\mathcal{X}}^k_{\lambda/\mu}(q)$.

Proof. The operators $h_k(\mathbf{u})$ commute and satisfy the ribbon Pieri rule (4.8) with respect to the basis $\{\lambda \mid \lambda \in \mathcal{P}\}$, by definition. The claim follows from Theorem 4.20 applied to $V = \mathbf{F}$ and $v_\lambda = \lambda$. □

The next theorem is a ribbon analogue of the classical Murnaghan-Nakayama rule which calculates the characters of the symmetric group.

Theorem 4.22 (Ribbon Murnaghan-Nakayama Rule). *Let $k \geq 1$ be an integer and ν be a partition with n-core δ. Then*

$$\left(1 + q^{2k} + \cdots + q^{2k(n-1)}\right) p_k \mathcal{G}_{\nu/\delta}(X;q) = \sum_{\mu} \widetilde{\mathcal{X}}^k_{\mu/\nu}(q) \mathcal{G}_{\mu/\delta}(X;q). \tag{4.10}$$

Also

$$k \frac{\partial}{\partial p_k} \mathcal{G}_{\nu/\delta}(X;q) = \sum_{\mu} \widetilde{\mathcal{X}}^k_{\nu/\mu}(q) \mathcal{G}_{\mu/\delta}(X;q).$$

Proof. The theorem follows from Theorems 4.20 and 4.12, where $V = \Lambda(q)$ and $v_\lambda = \mathcal{G}_{\lambda/\delta}(X;q)$. □

It is rather difficult to interpret Theorem 4.22 in terms of the n-quotient at $q = 1$. When $q = 1$ the product $\left(1 + q^{2k} + \cdots + q^{2k(n-1)}\right) p_k \mathcal{G}_{\lambda/\delta}(X;q)$ becomes $np_k s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}$ which may be written as the sum of n usual Murnaghan-Nakayama rules as

$$\sum_{i=0}^{n-1} s_{\lambda^{(0)}} \cdots \left(p_k s_{\lambda^{(i)}}\right) \cdots s_{\lambda^{(n-1)}}.$$

Thus we might expect that border ribbon strips of size k correspond to adding a usual ribbon strip of size k to one partition in the n-quotient. However, the following example shows that this cannot work.

Example 4.23. By the ribbon Murnaghan-Nakyama rule (Theorem 4.22) with $k = n = 2$ and $\nu = \emptyset$,

$$(1 + q^4) p_2 \cdot 1 = \mathcal{G}_{(4)} + q\mathcal{G}_{(3,1)} + (q^2 - 1)\mathcal{G}_{(2,2)} - q\mathcal{G}_{(2,1,1)} - q^2\mathcal{G}_{(1,1,1,1)}.$$

We can compute directly that

$$\mathcal{G}_{(4)} = h_2, \quad \mathcal{G}_{(3,1)} = qh_2, \quad \mathcal{G}_{(2,1,1)} = qe_2$$
$$\mathcal{G}_{(2,2)} = q^2h_2 + e_2, \quad \mathcal{G}_{(1,1,1,1)} = q^2e_2,$$

verifying Theorem 4.22 directly. On the other hand, the shapes which correspond to a single border strip in one partition of the 2-quotient are $\{(4),(3,1),(2,1,1),(1,1,1,1)\}$ and the corresponding $\mathcal{G}_\lambda$ terms do not give $(1+q^4)p_2$.

It seems possible that the ribbon Murnaghan-Nakayama rule may have some relationship with the representation theory of the wreath products $S_n \mathrm{wr} C_p$, or even more likely to the cyclotomic Hecke algebras associated to these wreath products (see for example [47]).

4.5 The ribbon involution ω_n

Define a semi-linear involution $v \mapsto v'$ on $\mathbf{F}$ by $q \mapsto q' = q^{-1}$ and

$$\lambda \longmapsto \lambda'.$$

Proposition 4.24 ([40, Proposition 7.10]). *For all $w \in \mathbf{F}$ and compositions β satisfying $|\beta| = k$ we have*

$$(h_\beta(\mathbf{u}) \cdot w)' = q^{-(n-1)k} e_\beta(\mathbf{u}) \cdot w', \qquad (h_\beta^\perp(\mathbf{u}) \cdot w)' = q^{-(n-1)k} e_\beta^\perp(\mathbf{u}) w'.$$

Proof. We use the descriptions of the action of $h_k(\mathbf{u})$ and $e_k(\mathbf{u})$ in terms of horizontal and vertical ribbon strips, together with the calculation $\mathrm{spin}(T) + \mathrm{spin}(T') = (n-1) \cdot r$ for a ribbon tableau T and its conjugate T' which contain r ribbons. ☐

Now we will define an involution w_n on $\Lambda(q)$ which is essentially the image of the involution $v \mapsto v'$ on the Fock space $\mathbf{F}$. However, this involution will turn out to be not just a semi-linear involution, but also a $\mathbb{Q}$-algebra isomorphism of $\Lambda(q)$.

Definition 4.25. Define the *ribbon involution* $w_n : \Lambda(q) \to \Lambda(q)$ as the semi-linear map satisfying $w_n(q) = q^{-1}$ and

$$w_n(s_\lambda) = q^{(n-1)|\lambda|} s_{\lambda'}.$$

Theorem 4.26. *The map w_n is a $\mathbb{Q}$-algebra homomorphism which is an involution. It maps $\mathcal{G}_{\lambda/\mu}$ into $\mathcal{G}_{(\lambda/\mu)'}$ for every skew shape λ/μ.*

Proof. The fact that w_n is an algebra homomorphism follows from the fact that if $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$ then $s_{\lambda'} s_{\mu'} = \sum c^\nu_{\lambda\mu} s_{\nu'}$, and that the grading is preserved by multiplication. That w_n is an involution is a quick calculation.

For the last statement, we use Proposition 4.24 and the fact that the involution $w(h_n) = e_n$ satisfies $w(s_\lambda) = s_{\lambda'}$ to obtain $(S_\nu \cdot \mu)' = q^{-(n-1)k} S_{\nu'} \cdot \mu'$. By Lemma 3.13 this implies that

$$\sum_\lambda c^\nu_{\lambda/\mu}(q^{-1})\lambda' = q^{-(n-1)k} \sum_\lambda c^{\nu'}_{\lambda'/\mu'}(q)\lambda'.$$

Here $k = |\nu|$. Equating coefficients of λ' we obtain $c^{\nu}_{\lambda/\mu}(q^{-1}) = q^{-(n-1)k} c^{\nu'}_{\lambda'/\mu'}(q)$. Thus

$$w_n(\mathcal{G}_{\lambda/\mu}) = \sum_{\nu} w_n(c^{\nu}_{\lambda/\mu}(q)s_\nu) = \sum_{\nu} \left(c^{\nu'}_{\lambda'/\mu'}(q)q^{-(n-1)|\nu|}\right) q^{(n-1)|\nu|} s_{\nu'} = \mathcal{G}_{\lambda'/\mu'}.$$

□

Let $\Upsilon_{q,n}$ denote the map $\Lambda(q) \to \Lambda(q)$ given by the plethysm $f \mapsto f[(1+q^2+\cdots+q^{2(n-1)})X]$. Note that $p_k[(1+q^2+\cdots+q^{2(n-1)})X] = (1+q^{2k}+\cdots+q^{2k(n-1)})p_k(X)$.

Proposition 4.27. *Let $f \in \Lambda(q)$ have degree k. Then we have*

$$q^{2(n-1)k}\omega_n\left(\Upsilon_{q,n}(f)\right) = \Upsilon_{q,n}\left(\omega_n(f)\right).$$

In particular, if $\lambda \vdash k$ we have

$$\omega_n\left(s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]\right) = q^{-(n-1)k} s_{\lambda'}[(1+q^2+\cdots+q^{2(n-1)})X].$$

Proof. Since both ω_n and $\Upsilon_{q,n}(f)$ are $\mathbb{Q}$-algebra homomorphisms we need only check this for the elements p_k and for q, for which the computation is straightforward. □

4.6 The ribbon Cauchy identity

Define the formal power series $\Omega_n(XY;q)$ and $\tilde{\Omega}_n(XY;q)$ by

$$\Omega_n(XY;q) = \prod_{i,j}\prod_{k=0}^{n-1}\frac{1}{1-x_iy_jq^{2k}} \quad ; \quad \tilde{\Omega}_n(XY;q) = \prod_{i,j}\prod_{k=0}^{n-1}\left(1+x_iy_jq^{2k}\right).$$

A combinatorial proof via ribbon insertion of the following identity was given by van Leeuwen [42].

Theorem 4.28 (Ribbon Cauchy Identity). *Fix n as usual and a n-core δ. Then*

$$\Omega_n(XY;q) = \sum \mathcal{G}_{\lambda/\delta}(X;q)\mathcal{G}_{\lambda/\delta}(Y;q)$$

where the sum is over all λ such that $\tilde{\lambda} = \delta$.

Unlike for the Schur functions, this does not imply that the $\mathcal{G}_{\lambda/\delta}$ form an orthonormal basis under a certain inner product, as they are not linearly independent.

Proof. By Corollary 4.11 we have

$$s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X] = \sum_{\mu\in\mathcal{P}_\delta} c^{\lambda}_{\mu/\delta}(q)\mathcal{G}_{\mu/\delta}(X;q).$$

Thus

$$\begin{aligned}\sum_{\lambda} s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X]s_\lambda(Y) &= \sum_{\mu\in\mathcal{P}_\delta}\left(\sum_{\lambda} c^{\lambda}_{\mu/\delta}(q)s_\lambda(Y)\right)\mathcal{G}_{\mu/\delta}(X;q)\\ &= \sum_{\mu\in\mathcal{P}_\delta}\mathcal{G}_{\mu/\delta}(X;q)\mathcal{G}_{\mu/\delta}(Y;q).\end{aligned}$$

Let $\Upsilon_{q,n}(X)$ denote the algebra automorphism of $\Lambda[X](q) \otimes_K \Lambda[Y](q)$ given by applying $\Upsilon_{q,n}$ to the X variables only. Applying $\Upsilon_{q,n}(X)$ to $\log\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \sum_k \frac{1}{n} p_k(X) p_k(Y)$ gives $\log\left(\prod_{i,j} \prod_{k=1}^{n-1} \frac{1}{1-x_i y_j q^{2k}}\right)$ which is exactly $\log(\Omega_n)$. Thus applying $\Upsilon_{q,n}(X)$ to the usual Cauchy identity for Schur functions ($\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(X) s_\lambda(Y)$) gives

$$\Omega_n(XY;q) = \sum_\lambda s_\lambda[(1+q^2+\cdots+q^{2(n-1)})X] s_\lambda(Y)$$

from which the Theorem follows. □

Theorem 4.28 can also be deduced directly from Theorem 3.6. See Section 5.2.3.

Now let us compute $\omega_n(\Omega)$ where we let $\omega_n : \Lambda[X](q) \otimes_K \Lambda[Y](q) \to \Lambda[X](q) \otimes_K \Lambda[Y](q)$ act on the X variables by

$$\omega_n(f(X;q) \otimes g(Y;q)) \longmapsto \omega_n(f(X;q)) \otimes g(Y;q^{-1}).$$

One checks immediately that this is indeed an algebra involution. We have (fixing an n-core δ)

$$\omega_n(\Omega_n) = \sum_{\lambda \in \mathcal{P}_\delta} \mathcal{G}_{\lambda'/\delta'}(X;q) \mathcal{G}_{\lambda/\delta}(Y;q^{-1}).$$

Also,

$$\omega_n(\Omega_n) = \sum_\lambda q^{(n-1)|\lambda|} s_{\lambda'}(X) s_\lambda[(1+q^{-2}+\cdots+q^{-2(n-1)})Y] = \prod_{i,j} \prod_{k=0}^{n-1} (1+x_i y_j q^{n-1-2k}).$$

Thus

$$\sum_{\lambda \in \mathcal{P}_\delta} \mathcal{G}_{\lambda'/\delta'}(X;q) \mathcal{G}_{\lambda/\delta}(Y;q^{-1}) = \prod_{i,j} \prod_{k=0}^{n-1} (1+x_i y_j q^{n-1-2k}).$$

If we multiply the d^{th} graded piece of each side by $q^{(n-1)d}$ we obtain the following result.

Proposition 4.29 (Dual Ribbon Cauchy Identity). *Fix an n-core δ. We have*

$$\tilde{\Omega}_n(XY;q) = \sum_{\lambda \in \mathcal{P}_\delta} q^{(n-1)|\lambda/\tilde{\lambda}|} \mathcal{G}_{\lambda'/\delta'}(X;q) \mathcal{G}_{\lambda/\delta}(Y;q^{-1}).$$

4.7 Skew and super ribbon functions

We now describe some properties of the skew ribbon functions $\mathcal{G}_{\lambda/\mu}(X;q)$. Unfortunately, we have been unable to describe them in analogy with the formula $s_{\lambda/\mu} = s_\lambda^\perp s_\mu$. However, we do have the following *skew ribbon Cauchy identity.*

Proposition 4.30. *Let μ be any partition. Then*

$$\mathcal{G}_{\mu/\tilde{\mu}}(X;q) \Omega_n(XY;q) = \sum_\lambda \mathcal{G}_{\lambda/\tilde{\lambda}}(X;q) \mathcal{G}_{\lambda/\mu}(Y;q)$$

where the sum is over all λ satisfying $\tilde{\lambda} = \tilde{\mu}$.

Proof. Lemma 3.13 and Theorem 4.10 imply that

$$s_\nu[(1+q^2+\ldots+q^{2(n-1)})X]\mathcal{G}_{\mu/\tilde{\mu}}(X;q)=\sum_\lambda c^\nu_{\lambda/\mu}(q)\mathcal{G}_{\lambda/\tilde{\lambda}}(X;q).$$

Now multiply both sides by $s_\nu(Y)$ and sum over ν. Finally use Theorem 4.28. □

Alternatively, one can prove Proposition 4.30 using Theorem 3.6 directly.

Schilling, Shimozono and White [51] have also used skew ribbon functions, as follows (their original result used cospin rather than spin). By the combinatorial definition of $\mathcal{G}_\lambda$ we immediately have the coproduct expansion $\mathcal{G}_\lambda(X+Y;q)=\sum_\mu \mathcal{G}_\mu(X;q)\mathcal{G}_{\lambda/\mu}(Y;q)$. Since (see [46]), $\Delta f=\sum_\mu s_\mu^\perp f\otimes s_\mu$ we get immediately that

$$s_\nu^\perp \mathcal{G}_\lambda(X;q)=\sum_\mu \mathcal{G}_\mu(X;q)\left\langle \mathcal{G}_{\lambda/\mu}(Y;q),s_\nu\right\rangle.$$

Setting $\nu=(k)$ we obtain the lowering version of the Pieri rule (Proposition 4.14).

Another related generalisation of the usual ribbon functions are super ribbon functions. Fix a total order $\prec$ on two alphabets $A=\{1<2<3<\cdots\}$ and $A'=\{1'<2'<3'<\cdots\}$ (which we assume to be compatible with each of their natural orders). For example, one could pick $1\prec 1'\prec 2\prec 2'\prec\cdots$.

Definition 4.31. A *super ribbon tableau* T of shape λ/μ is a ribbon tableau of the same shape with ribbons labelled by the two alphabets such that the ribbons labelled by a for $a\in A$ form a horizontal ribbon strip and those labelled by a' for $a'\in A'$ form a vertical ribbon strip. These strips are required to be compatible with the chosen total order. Thus the shape obtained by removing ribbons labelled by elements $\succ i$ must be a skew shape $\lambda_{\preceq i}/\mu$, for each $i\in A\cup A'$.

Define the *super ribbon function* $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ as the following generating function:

$$\mathcal{G}_{\lambda/\mu}(X/Y;q)=\sum_T q^{\mathrm{spin}(T)}x^{w(T)}(-y)^{w'(T)}$$

where the sum is over all super ribbon tableaux T of shape λ/μ and $w(T)$ is the weight in the first alphabet A while $w'(T)$ is the weight in the second alphabet A'. For a composition α, we use $(-y)^\alpha$ to stand for $(-y_1)^{\alpha_1}(-y_2)^{\alpha_2}\cdots(-y_l)^{\alpha_l}$.

Proposition 4.32. *The super ribbon function $\mathcal{G}_{\lambda/\mu}(X/Y;q)$ is a symmetric function in the X and Y variables, separately. It does not depend on the total order on the alphabets A and A'.*

Proof. If we pick the total order on $A\cup A'$ to be so that $a>a'$ for any $a\in A$ and $a'\in A'$ then we have $[x^\alpha(-y)^\beta]\mathcal{G}_{\lambda/\mu}(X/Y;q)=\langle h_\alpha(\mathbf{u})e_\beta(\mathbf{u})\mu,\lambda\rangle$ for any compositions α and β. The proof of symmetry is completely analogous to that of Theorem 3.12, using the commutativity of both the operators $\{h_k(\mathbf{u})\}$ and $\{e_k(\mathbf{u})\}$. The last statement requires the fact that $\{h_k(\mathbf{u})\}$ commutes with $\{e_k(\mathbf{u})\}$. □

4.8 The ribbon inner product and the bar involution on $\Lambda(q)$

Definition 4.33. Let $\langle .,. \rangle_n : \Lambda(q) \times \Lambda(q) \to K$ be the K-bilinear map defined by

$$\left\langle p_\lambda[(1+q^2+\cdots+q^{2(n-1)})X], p_\mu \right\rangle = z_\lambda \delta_{\lambda\mu}.$$

It is clear that $\langle .,. \rangle_n$ is non-degenerate. The inner product $\langle .,. \rangle_n$ is related to Ω_n in the same way as the usual inner product is related to the usual Cauchy kernel – the following claim is immediate.

Proposition 4.34. *Two bases $\{v_\lambda\}$ and $\{w_\lambda\}$ of $\Lambda(q)$ are dual with respect to $\langle .,. \rangle_n$ if and only if*

$$\sum_\lambda v_\lambda(X) w_\lambda(Y) = \Omega_n.$$

In particular, $\left\{ s_\lambda[(1+q^2+\ldots+q^{2(n-1)})X] \right\}$ is dual to $\{s_\lambda\}$.

Lemma 4.35. *The inner product $\langle .,. \rangle_n$ is symmetric.*

Proof. This is clear from the definition as we can just check this on the basis p_λ of $\Lambda(q)$. □

Recall that for $f \in \Lambda$, $f^\perp$ denotes its adjoint with respect to the Hall inner product.

Lemma 4.36. *The operator $f^\perp$ is adjoint to multiplication by $\Upsilon_{q,n}(f) \in \Lambda(q)$.*

Proof. This is a consequence of $\langle f, g \rangle = \langle \Upsilon_{q,n}(f), g \rangle_n$. □

The inner product $\langle .,. \rangle_n$ is compatible with the inner product $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$ on $\mathbf{F}$ when we restrict our attention to the space of highest weight vectors of $U_q(\widehat{\mathfrak{sl}}_n)$.

Proposition 4.37. *Let $u, v \in \mathbf{F}$ be highest weight vectors for the action of $U_q(\widehat{\mathfrak{sl}}_n)$. Then $\langle \mathbf{\Phi}(u), \mathbf{\Phi}(v) \rangle_n = \langle u, v \rangle$.*

Proof. We check the claim for the basis $\{B_{-\lambda} \cdot \emptyset\}$ of the space of highest weight vectors in $\mathbf{F}$. □

The bar involution $^- : \mathbf{F} \to \mathbf{F}$ of Section 4.1.3 also has an image under $\mathbf{\Phi}$.

Definition 4.38. Define the $\mathbb{Q}$-algebra involution $^- : \Lambda(q) \to \Lambda(q)$ by $\overline{q} = q^{-1}$ and

$$p_k \longmapsto q^{2(n-1)k} p_k.$$

It is clear that $^-$ is indeed an involution. We have the following basic properties of $^-$, imitating similar properties of the bar involution of $\mathbf{F}$ ([40, Theorem 7.11]).

Proposition 4.39. *Let $u, v \in \Lambda(q)$. The involution $^- : \Lambda(q) \to \Lambda(q)$ has the following properties:*

$$\overline{\mathbf{\Phi}(G_{n\lambda})} = \mathbf{\Phi}(G_{n\lambda}),$$
$$\overline{\Upsilon_{q,n}(p_k)} = \Upsilon_{q,n}(p_k),$$
$$\langle \overline{u}, v \rangle_n = \left\langle \omega_n(u), \overline{\omega_n(v)} \right\rangle_n.$$

Proof. As $^-$ is an algebra homomorphism, the first statement follows from the second statement and Theorem 4.10. The second statement is a straightforward computation. For the last statement, we compute explicitly both sides for the basis p_λ of $\Lambda(q)$. □

Proposition 4.39 and Theorem 4.6 show that $\overline{\mathbf{\Phi}(v)} = \mathbf{\Phi}(\overline{v})$ for all u, v in the subspace of highest weight vectors in $\mathbf{F}$. However this is not true in general. For example, $(3,1) + q(2,2) + q^2(2,1,1)$ is bar invariant in $\mathbf{F}$ but its image under $\mathbf{\Phi}$ is not.

4.9 Ribbon insertion

In this section we put the ribbon Pieri formula (Theorem 4.12) and ribbon Cauchy identity (Theorem 4.28) in the context of ribbon Robinson-Schensted-Knuth (RSK) insertion (following partly [54, 32]). We will give a complete proof of both for the case $n = 2$. Van Leeuwen [42] has recently described a spin-preserving RSK correspondence which gives a combinatorial proof of the ribbon Cauchy identity for general n. Unfortunately, the correspondence does not involve insertion and hence does not seem to lead to a proof of the ribbon Pieri rule.

4.9.1 General ribbon insertion

In what follows, we assume familiarity with the usual RSK insertion and use the usual language of the subject (see [55]). We will assume that all ribbon tableaux have shapes with empty n-core for simplicity, though everything can be generalised to the general case. A *biletter* is a triple (c_k, i_k, j_k) where c_i is the *color* taking values in $\{0, \ldots, n-1\}$ and the i_k, j_k are positive integers. Define the *weight* of a biletter (c_k, i_k, j_k) to be $w((c_k, i_k, j_k)) = q^{2c_k} y_{i_k} x_{j_k}$. The weight $w(\mathbf{w})$ of a multiset of biletters $\mathbf{w}$ is then the sum of the weights of the biletters. The weight generating function for multisets of biletters is

$$\sum_{\mathbf{w}} w(\mathbf{w}) = \prod_{i,j} \prod_{k=0}^{n-1} \frac{1}{1 - x_i y_j q^k}. \tag{4.11}$$

A *ribbon Schensted bijection* $\pi : \mathbf{w} \mapsto (P_r(\mathbf{w}), Q_r(\mathbf{w}))$ is a bijection between multisets of biletters and pairs of ribbon tableaux of the same shape which is weight preserving, where the weight of a P-tableau is $q^{\mathrm{spin}(P)} x^{w(P)}$ and the weight of a Q-tableau is $q^{\mathrm{spin}(Q)} y^{w(Q)}$. Van Leeuwen [42] has given exactly such a bijection. By (4.11), a ribbon Schensted bijection results immediately in a proof of the ribbon Cauchy identity (Theorem 4.28).

Suppose now that the bijection π is defined recursively via insertion of ribbons (c, j) (a ribbon labelled j of spin c) into a tableau T:

$$P_r(\mathbf{w}) = ((\cdots((\emptyset \longleftarrow (c_1, j_1)) \longleftarrow (c_2, j_2)) \cdots) \longleftarrow (c_m, j_m)).$$

Given a tableau T, the tableau $T' = T \leftarrow (c, j)$ should satisfy (a) the tableau T' has one more ribbon than T, labelled j, (b) $\mathrm{sh}(T')/\mathrm{sh}(T)$ is a skew shape which is a ribbon, and (c) $\mathrm{spin}(T') + \mathrm{spin}(\mathrm{sh}(T')/\mathrm{sh}(T)) = \mathrm{spin}(T) + 2c$. Let

$$T' = T \longleftarrow (c, j)\ ;\ T'' = T' \longleftarrow (c', j').$$

We will say that the insertion $T \leftarrow (c, j)$ has the *ribbon increasing property* if the ribbon $\mathrm{sh}(T')/\mathrm{sh}(T)$ lies to the left of $\mathrm{sh}(T'')/\mathrm{sh}(T')$ if and only if $(c, j) \leq (c', j')$. Here $\leq$ should be some total order on labelled ribbons (c, j).

Fix a ribbon tableau T. Then we can construct a (weight-preserving) bijection between multisets of ribbons $\{(c_i, j_i)\}$ of size k and ribbon tableaux T' such that $\mathrm{sh}(T')/\mathrm{sh}(T)$ is a horizontal ribbon strip of length k, as follows:

$$T' = ((\cdots((T \longleftarrow (c_1, j_1)) \longleftarrow (c_2, j_2)) \cdots) \longleftarrow (c_k, j_k)).$$

The ribbons (c_i, j_i) are inserted according to the order $<$ thus ensuring the resulting shape changes by a horizontal ribbon strip. Thus:

Observation 4.40. *Suppose $\pi : \mathbf{w} \mapsto (P_r(\mathbf{w}), Q_r(\mathbf{w}))$ is a ribbon Schensted bijection defined by the insertion "$\leftarrow$", satisfying a ribbon increasing property. Then π leads to a combinatorial proof of the ribbon Pieri formula (Theorem 4.12).*

The generating function $H(t)$ of Section 4.3 can be interpreted as the weight generating function of ribbons (c, j) with weight $w(c, j) = q^{2k} x_j$. For the $k = 1$ case of the ribbon Pieri rule, no ribbon increasing property is required, only that the bijection π is given by some kind of insertion algorithm. Shimozono and White have such an insertion algorithm without a ribbon increasing property.

Proposition 4.41. *Shimozono and White's spin-preserving ribbon insertion leads to a combinatorial proof that*

$$(1 + q^2 + \ldots + q^{2(n-1)}) h_1(X) \mathcal{G}_\lambda(X; q) = \sum_{\mu} q^{\mathrm{spin}(\mu/\lambda)} \mathcal{G}_\mu(X; q)$$

where the sum is over all μ such that μ/λ is a n-ribbon.

We will not give the details of Shimozono and White's ribbon insertion here but refer the reader to the paper [54]. They give an "insertion" algorithm which inserts a ribbon (c, j) with particular spin into a semistandard ribbon tableau T. Roughly speaking, this ribbon insertion is determined by forcing all ribbons to bump by rows to another ribbon of the same spin. It is possible however to insist that all ribbons with certain spins bump by columns instead. Unfortunately, it appears that none of these algorithms have a ribbon increasing property.

4.9.2 Domino insertion

Observation 4.40 becomes a proof for the case $n = 2$, where a ribbon is in fact a *domino*. In [53], Shimozono and White, extending work of Garfinkle [13] and Barbasch and Vogan [1], gave a domino Schensted bijection. This was extended to the case of non-empty 2-core in [32]:

Theorem 4.42. *Fix a 2-core δ. There is a bijection between colored biwords $\mathbf{w}$ of length m with two colors $\{0, 1\}$ and pairs $(P_d(\mathbf{w}), Q_d(\mathbf{w}))$ of semistandard domino tableaux with the same shape $\lambda \in \mathcal{P}_\delta$ and $|\lambda| = 2m + |\delta|$ which is weight-preserving.*

We will describe this bijection for the standard case. In brief, domino insertion is determined by insisting that horizontal dominoes bump by rows and vertical dominoes bump by columns. More precisely, let S be a domino tableau with no value repeated (but still semistandard), and i some number not used in S. We will describe $(S \leftarrow (0, i))$ and

$(S \leftarrow (1,i))$ which correspond to the insertion of a horizontal (color 0) and vertical domino (color 1) labelled i respectively.

Let $T_{<i}$ be the sub-tableau of S consisting of all dominoes labelled with numbers less than i. Then set $T_{\leq i}$ to be $T_{<i}$ union a horizontal domino at the end of the first row labelled i or a vertical domino in the first column labelled i depending on what we are inserting. Now for $j > i$ we will recursively define $T_{\leq j}$ given $T_{\leq j-1}$. If there is no domino labelled j in T then $T_{\leq j} = T_{\leq j-1}$. Otherwise let γ_j denote the domino labelled j in S and set $\lambda = \mathrm{sh}(T_{\leq j-1})$. We distinguish four cases.

1. If $\gamma_j \cap \lambda = \emptyset$ then set $T_{\leq j} = T_{\leq j-1} \cup \gamma_j$.

2. If $\gamma_j \cap \lambda = \gamma_j$ is a horizontal domino in row k then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a horizontal domino labelled j to row $k+1$.

3. If $\gamma_j \cap \lambda = \gamma_j$ is a vertical domino in column k then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a vertical domino labelled j to column $k+1$.

4. If $\gamma_j \cap \lambda = (l,m)$ is a single square then $T_{\leq j}$ is obtained from $T_{\leq j-1}$ by adding a domino labelled j so that the total shape of $T_{\leq j}$ is $\lambda \cup (l+1, m+1)$.

The resulting tableau $T_{<\infty} = (S \leftarrow (c,i))$. Figure 4-2 gives an example of domino insertion.

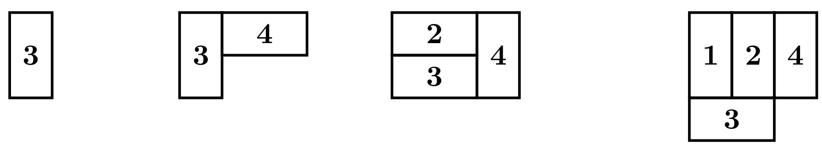

Figure 4-2: The result of the insertion $((((\emptyset \leftarrow (1,3)) \leftarrow (0,4)) \leftarrow (0,2)) \leftarrow (1,1))$.

It turns out that domino insertion has a domino increasing property. This was first shown by Shimozono and White [54] by connecting domino insertion with mixed insertion. [32] gives a different proof using growth diagrams. The domino increasing property can be described by specifying an order $<$ on dominoes as follows (γ_i denotes a domino labelled i)

1. If γ_i is horizontal and γ_j vertical then $\gamma_i > \gamma_j$.

2. If γ_i and γ_j are both horizontal then $\gamma_i > \gamma_j$ if and only if $i > j$.

3. If γ_i and γ_j are both vertical then $\gamma_i > \gamma_j$ if and only if $i < j$.

Under this order, domino insertion has a ribbon increasing property, as described in Section 4.9.1.

Lemma 4.43 ([53, 32]). *Let T be a domino tableaux without the labels i and j. Set $T' = (T \leftarrow \gamma_i)$ and $T'' = (T' \leftarrow \gamma_j)$ for some dominoes γ_i and γ_j. Then $\mathrm{sh}(T'/T)$ lies to the left of $\mathrm{sh}(T''/T')$ if and only if $\gamma_i < \gamma_j$.*

This increasing property is retained when the bijection is extended to the semistandard case. Using Observation 4.40, we obtain:

Proposition 4.44. *Semistandard domino insertion proves the domino Pieri rule (Theorem 4.12 for $n = 2$).*

4.10 Some Open Questions

Cospin vs. spin. Recall that $\mathrm{cosp}(T) = \mathrm{mspin}(T) - \mathrm{spin}(T)$ for a ribbon tableau T where mspin is the maximum spin for a ribbon tableau with of the same shape. It is easy to see that $\mathrm{cosp}(T)$ is always even. In many situations it appears that the cospin statistic is more natural than the statistic spin. For example, Lascoux, Leclerc and Thibon [38] have shown that the cospin $\widetilde{\mathcal{H}}(X;q)$ functions are generalisations of Hall-Littlewood functions. Cospin also appears to be the natural statistic when finding connections between ribbon tableaux and rigged configurations (see for example [49]).

All the formulae in this Chapter can be phrased in terms of cospin if suitable powers of q are inserted. However, it is clear that the formulae presented in terms of spin is the more natural form.

Jacobi-Trudi and alternant formulae. The ribbon Pieri rule (Theorem 4.12) we have given stops short of giving a closed formula for the functions $\mathcal{G}_\lambda(X;q)$. It is well known that the Schur functions can be written as $s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$ and as $s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^{l(\lambda)}$. Most algebraic treatments (see [46]) of the theory of symmetric functions use these formulae as the basis of all the algebraic computations for Schur functions. It would be nice to have a similar closed formula for the ribbon functions.

Other "Pieri" and "Littlewood-Richardson" rules. In [31], a "half"-Pieri rule was given for the case $n = 2$ which described the product $h_k(X)\mathcal{G}_\lambda^{(2)}(X;q)$ for certain partitions λ. It would be interesting to give rules for the products $h_k(X)\mathcal{G}_\lambda(X;q)$, $s_\mu(X)\mathcal{G}_\lambda(X;q)$ and $\mathcal{G}_\lambda(X;q)\mathcal{G}_\mu(X;q)$ for all n.

Enumerative problems. Stanley [55] has given a "hook content formula" for the specialisation $s_\lambda(1, t, t^2, \ldots, t^r)$ of the Schur functions. In particular this gives the hook length formula for the number of standard Young tableaux of a particular shape. At $q = 1$ the corresponding problem for ribbon tableaux is trivial due to Littlewood's n-quotient map. However, can anything be done for arbitrary q?

When $n = 2$, the specialisation $q^2 = -1$ relates domino tableaux to the study of enumerative study of sign-imbalance [56, 61, 32]. It is not clear whether this can be generalised to arbitrary n.

Graded S_n representations and $\mathbf{h}_k$. The non-negativity of the q-Littlewood Richardson coefficients $c_\lambda^\mu(q)$ would follow from the existence of a graded S_n representation with Frobenius character $\mathcal{G}_\lambda(X;q)$ (where the coefficient of powers of q correspond to the graded parts).

Such a graded S_n representation can be easily described for the ribbon homogeneous function $\mathbf{h}_k$ (see [55, Ex. 7.75]). Let S_k act on the multiset $M = \{1^{n-1}, 2^{n-1}, \ldots, k^{n-1}\}$ in the natural way. Then the representation corresponding to $\mathbf{h}_k$ is given by the action of S_k on the subsets of M with the grading given by the size of such a subset. This suggests that one might seek subrepresentations of this representation which correspond to the $\mathcal{G}_\lambda(X;q)$ for $l(\lambda) \leq n$ (see Equation (4.5)).

Generating series for ribbon functions. In [31], Kirillov, Lascoux, Leclerc and Thibon gave a number of generating functions for domino functions which were subsequently generalised in [32]. As a special case, we have the following product expansion for $n = 2$:

$$\sum_\lambda \mathcal{G}_\lambda^{(2)}(X;q) = \frac{\prod_i (1 + q x_i)}{\prod_i (1 - x_i) \prod_i (1 - q^2 x_i^2) \prod_{i<j} (1 - x_i x_j) \prod_{i<j} (1 - q^2 x_i x_j)}.$$

Can this be generalised to other values of n?

Other incarnations of Λ. Often the Fock space $\mathbf{F}$ is identified with Λ via

$$\lambda \longleftrightarrow s_\lambda.$$

This gives $\mathbf{F}$ the extra structure of an algebra. In this context, our map $\mathbf{\Phi}$ can be considered to be an operator from $\Lambda(q)$ to $\Lambda(q)$. In the notation of [38], $\mathbf{\Phi}$ would be the adjoint ϕ_q of the operator p_n^q which sends h_α to $h_\alpha(\mathbf{u}) \cdot 1$.

Leclerc [39] has studied another embedding $\iota : \Lambda \to \mathbf{F}$ given by $p_\lambda \mapsto B_{-\lambda} \cdot \emptyset$. Altering this slightly, we may define a K-linear embedding $\iota_q : \Lambda(q) \to \mathbf{F}$ given by $\Upsilon_{q,n}(p_\lambda) \mapsto B_{-\lambda} \cdot \emptyset$. By Theorem 4.10, we see that the composition $\mathbf{\Phi} \circ \iota_q : \Lambda(q) \to \Lambda(q)$ is the identity. Leclerc has connected ι with the Macdonald polynomials and it is likely that our setup can be connected with many other aspects of symmetric function theory in this way.

Chapter 5

Combinatorial generalization of the Boson-Fermion correspondence

Let $\{F_\lambda(X) \in \Lambda_K : \lambda \in S\}$ be a family of symmetric functions with coefficients in a field K (usually $\mathbb{Q}$, $\mathbb{Q}(q)$ or $\mathbb{Q}(q,t)$), where S is some indexing set. Many important families of symmetric functions have the following trio of properties.

1. They can be expressed as the generating functions for a "tableaux"-like set of objects:

$$F_\lambda(X) = \sum_T \mathrm{wt}_T x^T$$

   where the sum is over tableaux T with "shape" λ and we have weights $\mathrm{wt}_T \in K$ and x^T is a monomial in the x_i. Often the set S has a poset structure $(S, <)$ where each pair (λ, μ) such that $\lambda < \mu$ has been given a weighting $\mathrm{wt}(\lambda, \mu) \in K$. Then the tableaux are chains $T = (\lambda^{(0)} < \lambda^{(1)} < \cdots < \lambda^{(r)})$ in S with weighting $\mathrm{wt}_T = \prod_{i=1}^r \mathrm{wt}(\lambda^{(i-1)}, \lambda^{(i)})$.

2. Together with a dual family $\{G_\lambda(X) : \lambda \in S\}$ of symmetric functions, they satisfy a "Cauchy"-style identity:

$$\sum_{\lambda \in S} F_\lambda(X) G_\lambda(Y) = \prod_{i,j=1}^{\infty} \left(b_0 + b_1 x_i y_j + b_2 (x_i y_j)^2 + \cdots\right)$$

   where the coefficients $b_i \in K$.

3. They satisfy a "Pieri"-style formula:

$$\tilde{h}_k(X) F_\lambda(X) = \sum_{\mu \rightharpoonup_k \lambda} b_{\lambda,\mu} F_\mu(X)$$

   where $k \in \mathbb{Z}$ is a positive integer, $\{\tilde{h}_1, \tilde{h}_2, \ldots\} \in \Lambda_K$ is a sequence of symmetric functions and $b_{\lambda,\mu} \in K$ are coefficients for each pair λ, μ satisfying some condition $\mu \rightharpoonup_k \lambda$. The condition $\mu \rightharpoonup_k \lambda$ often involves the same structure as for the tableaux definition. For example, often one has $\mu \rightharpoonup_k \lambda$ for some $k \in \mathbb{Z}$ if and only if $\lambda < \mu$ and $\mathrm{wt}(\lambda, \mu) \neq 0$. In fact one often has the equality $\mathrm{wt}(\lambda, \mu) = b_{\lambda,\mu}$.

The simplest case is $F_\lambda = s_\lambda$, the family of Schur functions. The indexing set $S = \mathcal{P}$ is the set of partitions. The tableaux are usual semi-standard Young tableaux; that is, chains of partitions. The dual family $\{G_\lambda = s_\lambda\}$ is equal to the Schur functions again and in the Cauchy formula, all the coefficients $a_i = 1$. In the Pieri formula, $\tilde{h}_k = h_k$ are the homogeneous symmetric functions. The condition $\mu \rightharpoonup_k \lambda$ is that μ/λ is a horizontal strip of size k and all the coefficients $b_{\lambda,\mu} = 1$.

Other examples include the shifted Schur functions, Hall-Littlewood functions, Macdonald polynomials, and LLT's ribbon functions studied earlier in the thesis. In all these cases the indexing set S is in addition graded, and all three properties are compatible with this grading.

We shall give an explanation of this phenomenon by relating these symmetric functions to representations of Heisenberg algebras. As we shall see, our work can be considered a combinatorial generalisation of the classical *Boson-Fermion correspondence*.

5.1 The classical Boson-Fermion correspondence

Let K be a field with characteristic 0. In this chapter, the Heisenberg algebra H is the associative algebra over K with 1 generated by $\{B_k : k \in \mathbb{Z}\backslash\{0\}\}$ satisfying

$$[B_k, B_l] = l \cdot a_l \cdot \delta_{k,-l}$$

for some non-zero parameters $a_l \in K$ satisfying $a_l = -a_{-l}$. As an abstract algebra, H does not depend on the choice of the elements a_l, since the generators B_k can be re-scaled to force $a_l = 1$. However, we shall be concerned with representations of H, and some choices of the generators B_k will be more natural. Let $K[B_{-1}, B_{-2}, \ldots]$ denote the Bosonic Fock space representation of H, as in Section 4.1.1. One can identify $K[H_-] = K[B_{-1}, B_{-2}, \ldots]$ with the algebra Λ_K of symmetric functions over K by identifying B_{-k} with $a_k p_k$ for $k > 0$.

If V is a representation of H, then a vector $v \in V$ is called a *highest weight vector* if $B_k \cdot v = 0$ for $k > 0$. The following result is well known. See for example [21, Proposition 2.1].

Proposition 5.1. *Let V be an irreducible representation of H with non-zero highest weight vector $v \in V$. Then there exists a unique isomorphism of H-modules $V \xrightarrow{\sim} K[B_{-1}, B_{-2}, \ldots]$ sending $v \mapsto 1$.*

For the remainder of this section we assume that H is given by the parameters $a_l = 1$ for $l \geq 1$ and $a_l = -1$ for $l \leq -1$. Let $W = \oplus_{j\in\mathbb{Z}} K v_j$ be an infinite-dimensional vector space with basis $\{v_j : j \in Z\}$. Let $\mathcal{F}^{(0)}$ denote the vector space with basis given by semi-infinite monomials of the form $v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ where the indices satisfy:

(i) $i_0 > i_{-1} > i_{-2} > \cdots$

(ii) $i_k = k$ for k sufficiently small.

We will call $\mathcal{F}^{(0)}$ the *Fermionic Fock space*. Note that usually $\mathcal{F}^{(0)}$ is considered a subspace of a larger space $\mathcal{F} = \oplus_{m\in\mathbb{Z}} \mathcal{F}^{(m)}$. The spaces $\mathcal{F}^{(m)}$ are defined as for $\mathcal{F}^{(0)}$ with the condition (ii) replaced by the condition (ii$^{(m)}$): $i_k = k - m$ for k sufficiently small. Define an action of H on $\mathcal{F}^{(0)}$ by

$$B_k \cdot (v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = \sum_{j<0} v_{i_0} \wedge v_{i_{-1}} \wedge \cdots \wedge v_{i_{j-1}} \wedge v_{i_j - k} \wedge v_{i_{j+1}} \wedge \cdots . \qquad (5.1)$$

The monomials are to be reordered according to the usual exterior algebra commutation rules so that $v_{i_0} \wedge \cdots \wedge v_{i_j} \wedge v_{i_{j+1}} \wedge \cdots = -v_{i_0} \wedge \cdots \wedge v_{i_{j+1}} \wedge v_{i_j} \wedge \cdots$. Thus the sum on the right hand side of (5.1) is actually finite so the action is well defined. One can check that we indeed do obtain an action of H.

It is not hard to see that the representation of H on $\mathcal{F}^{(0)}$ is irreducible. The space $\mathcal{F}^{(0)}$ can be identified with $\mathbf{F}$ earlier in the thesis by $(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) \leftrightarrow (i_0, i_{-1}+1, \ldots)$.

The vector $\bar{v} = v_0 \wedge v_{-1} \wedge \cdots \in \mathcal{F}^{(0)}$ is a highest weight vector for this action of H. By Proposition 5.1, there exists an isomorphism $\sigma : \mathcal{F}^{(0)} \to \Lambda_K$ sending $\bar{v} \mapsto 1$. An algebraic version of the *Boson-Fermion correspondence* identifies the image of $v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ under the isomorphism σ.

Theorem 5.2 ([21, Lecture 6]). *Let $\lambda_k = i_{-k} + k$. Then $\sigma(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = s_\lambda$.*

In [21], this is called the "second" part of the boson-fermion correspondence. It is important in the study of a family of non-linear differential equations known as the *Kadomtzev-Petviashvili (KP) Hierarchy*. The "first" part consists of identifying the image of certain *vertex operators* under σ. The relationship between vertex operators and symmetric function theory have been studied previously in [19, 20, 46].

Our aim will be to generalise Theorem 5.2 to representations of Heisenberg algebras with arbitrary parameters $a_i \in K$. We will see that the symmetric functions that one obtains in this manner will always have a tableaux-like definition and satisfy Pieri and Cauchy identities. In our approach, we have ignored the vertex operators, but it would be interesting to see how they are related to our results.

5.2 The main theorem

5.2.1 Symmetric functions from representations of Heisenberg algebras

Let H be a Heisenberg algebra with parameters $a_i \in K$. Define $B_\lambda := B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_{l(\lambda)}}$ and let $D_k := \sum_{\lambda \vdash k} z_\lambda^{-1} B_\lambda$ where z_λ is as defined in Section 2.2.1. The elements $B_\lambda, D_k \in H$ are related in the same way as the elements $p_\lambda, h_k \in \Lambda$. Similarly define $B_{-\lambda} := B_{-\lambda_1} B_{-\lambda_2} \cdots B_{-\lambda_{l(\lambda)}}$ and $U_k := \sum_{\lambda \vdash k} z_\lambda^{-1} B_{-\lambda}$. Also let $S_\lambda \in H$ be given by $S_\lambda := \sum_\mu z_\mu^{-1} \chi_\mu^\lambda B_{-\mu}$ where the coefficients χ_μ^λ are the characters of the symmetric group given by $s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu$.

Let V be a representation of H with distinguished basis $\{v_s : s \in S\}$ for some indexing set S. For simplicity we will assume that both V and S are $\mathbb{Z}$-graded so that $v_s \in V$ are homogeneous elements and $\deg(v_s) = \deg(s)$, and that each graded component of V is finite-dimensional. We will also assume that the action of H is graded in the sense that $\deg(B_k) = -mk$ for some $m \in \mathbb{Z} \backslash \{0\}$. Define an inner product $\langle ., . \rangle : V \times V \to K$ on V by requiring that $\{v_s \mid s \in S\}$ forms an orthonormal basis, so that $\langle v_s, v_s' \rangle = \delta_{ss'}$.

Let $s, t \in S$. Define the generating functions

$$F_{s/t}^V(X) = F_{s/t}(X) := \sum_\alpha x^\alpha \left\langle U_{\alpha_l} U_{\alpha_{l-1}} \cdots U_{\alpha_1} \cdot t, s \right\rangle$$

where the sum is over all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$. Similarly define

$$G_{s/t}^V(X) = G_{s/t}(X) = \sum_\alpha x^\alpha \left\langle D_{\alpha_l} D_{\alpha_{l-1}} \cdots D_{\alpha_1} \cdot s, t \right\rangle .$$

Note that $F_{s/t}$ and $G_{s/t}$ are homogeneous with degree $\frac{\deg(s)-\deg(t)}{m}$. So in particular if $\frac{\deg(s)-\deg(t)}{m}$ is negative or non-integral then the generating functions are 0. For convenience we let $U_\alpha := U_{\alpha_l} U_{\alpha_{l-1}} \cdots U_{\alpha_1}$ and $D_\alpha := D_{\alpha_l} D_{\alpha_{l-1}} \cdots D_{\alpha_1}$.

The following Proposition is immediate from the definition, since U_k commutes with U_l and D_k commutes with D_l for all $k, l \in \mathbb{N}$.

Proposition 5.3. *The generating functions $F_{s/t}$ and $G_{s/t}$ are symmetric functions.*

As before, let $K[H_-] \subset H$ denote the subalgebra of H generated by $\{B_k \mid k < 0\}$ and similarly define $K[H_+] \subset H$. The definitions of $F_{s/t}$ and $G_{s/t}$ can be rephrased in terms of the *Heisenberg-Cauchy elements* $\Omega(H_-, X)$ and $\Omega(H_+, X)$ which lie in the completed tensor products $K[H_-]\hat{\otimes}\Lambda_K(X)$ and $K[H_+]\hat{\otimes}\Lambda_K(X)$ respectively:

$$\Omega(H_-, X) := \sum_\lambda U_\lambda \otimes m_\lambda(X) = \sum_\lambda z_\lambda^{-1} B_{-\lambda} \otimes p_\lambda = \sum_\lambda S_\lambda \otimes s_\lambda(X).$$

The last two equalities follow from the classical Cauchy identity. Also define $\Omega(H_+, X) \in K[H_+]\hat{\otimes}\Lambda_K(X)$ by $\Omega(H_+, X) = \sum_\lambda D_\lambda \otimes m_\lambda(X)$.

Thus for example, one has

$$F_{s/t}(X) = \langle \Omega(H_-, X) \cdot v_t, v_s \rangle$$

and

$$G_{s/t}(X) = \langle \Omega(H_+, X) \cdot v_s, v_t \rangle .$$

For example, one has in particular

$$G_{s/t}(X) = \sum_\lambda z_\lambda^{-1} p_\lambda \langle B_\lambda \cdot v_s, v_t \rangle . \tag{5.2}$$

Now let $b \in S$ be such that v_b is a highest weight vector for H. We will write $F_s := F_{s/b}$ and $G_s := G_{s/b}$. The element $\Omega(H_-, X) \cdot v_b \in V\hat{\otimes}\Lambda_K(X)$ depends only on the choice of v_b. The symmetric functions F_s are the coefficients of $\Omega(H_-, X) \cdot v_b$ when it is written in the basis $\{v_s \mid s \in S\}$:

$$\Omega(H_-, X) \cdot v_b = \sum_s v_s \otimes F_s(X).$$

5.2.2 Generalisation of Boson-Fermion correspondence

Let us suppose that $b \in S$ has been picked so that $v_b \in V$ is a highest weight vector for H. By Proposition 5.1, there is a canonical map of H-modules $H \cdot b \to \Lambda_K$ sending $v_b \mapsto 1$. Our choice of inner product for V allows us to give a map $V \to \Lambda_K$.

Theorem 5.4 (Generalised Boson-Fermion correspondence). *The map $\mathbf{\Phi} : V \to \Lambda_K$ given by $v_s \mapsto G_s(X)$ is a map of H-modules.*

Recall that B_{-k} acts on Λ_K by multiplication by $a_k p_k$ and B_k acts as $k\frac{\partial}{\partial p_k}$, for $k \geq 1$.

Proof. Let us calculate $B_l \cdot G_s$ and compare with $\mathbf{\Phi}(B_l \cdot v_s)$. Suppose first that $l < 0$ and let $k = -l$. Let λ be a partition and let μ be λ with one less part equal to k. If λ has no part equal to k, then μ can be any partition in the following formulae. First write $\langle B_\lambda B_l \cdot v_s, v_b \rangle = k a_k m_k(\lambda) \langle B_\mu v_s, v_b \rangle$, using a slight variation of Lemma 4.1 for our H. Alternatively, one can also compute

$$B_\lambda B_l \cdot v_s = B_\lambda \sum_c \langle B_l \cdot v_s, v_c \rangle v_c = \sum_{c,d} \langle B_l \cdot v_s, v_c \rangle \langle B_\lambda \cdot v_c, v_d \rangle v_d$$

so that taking the coefficient of v_b we obtain

$$k a_k m_k(\lambda) \langle B_\mu \cdot v_s, v_b \rangle = \sum_c \langle B_l \cdot v_s, v_c \rangle \langle B_\lambda \cdot v_c, v_b \rangle . \tag{5.3}$$

Now,

$$\begin{aligned} B_l \cdot G_s &= a_k p_k G_s \\ &= a_k \sum_\mu z_\mu^{-1} p_k p_\mu \langle B_\mu \cdot v_s, v_b \rangle && \text{by Equation (5.2),} \\ &= \sum_\lambda z_\lambda^{-1} p_\lambda \left(\sum_c \langle B_l \cdot v_s, v_c \rangle \langle B_\lambda \cdot v_c, v_b \rangle \right) && \text{using (5.3)} \\ &= \sum_c \langle B_l \cdot v_s, v_c \rangle \left(\sum_\lambda z_\lambda^{-1} \langle B_\lambda \cdot v_c, v_b \rangle \right) \\ &= \sum_c \langle B_l \cdot v_s, v_c \rangle G_c. \end{aligned}$$

This shows that $\mathbf{\Phi}(B_l \cdot v_s) = B_l \cdot \mathbf{\Phi}(v_s)$ for $l < 0$.

Now suppose $k > 0$, and let λ and μ be related as before. Then

$$\begin{aligned} B_k \cdot G_s &= k \sum_\lambda z_\lambda^{-1} \frac{\partial}{\partial p_k} p_\lambda \langle B_\lambda \cdot v_s, v_b \rangle \\ &= k \sum_\lambda z_\lambda^{-1} m_k(\lambda) p_\mu \langle B_\mu B_k \cdot v_s, v_b \rangle \\ &= \sum_\mu z_\mu^{-1} p_\mu \left\langle B_\mu \cdot \sum_c \langle B_k \cdot v_s, v_c \rangle v_c, v_b \right\rangle \\ &= \sum_c \langle B_k \cdot v_s, v_c \rangle \left(\sum_\mu z_\mu^{-1} p_\mu \langle B_\mu \cdot v_c, v_b \rangle \right) \\ &= \sum_c \langle B_k \cdot v_s, v_c \rangle G_c. \end{aligned}$$

This completes the proof. □

When V is irreducible, this map of H-modules does not depend on the choice of basis, but does depend on v_b. Since the degree $\deg(v_b)$ part of V is one dimensional, the image of $v \in V$ is given by the coefficient of the degree $\deg(v_b)$ part of $\Omega(H_+, X) \cdot v$.

If V is not irreducible then the map depends on the inner product $\langle ., . \rangle$ (or equivalently, the choice of orthonormal basis).

Note that a different action of H on Λ_K will allow us to replace the family G_s in Theorem 5.4 by F_s. More precisely, one can define the adjoint action ϑ of H on V by letting the generators B_k satisfy $\langle B_{-k} \cdot v_s, v_{s'} \rangle = \langle v_s, \vartheta(B_k) \cdot v_{s'} \rangle$. With this new representation of H on V, the roles of G_s and F_s are reversed.

5.2.3 Pieri and Cauchy identities

Let $h_k[a_i]$ denote the image of h_k under the map $\Lambda \to \Lambda_K$ given by $p_k \mapsto a_k p_k$. Let $h_k^\perp$ be the linear operator on Λ_K which is adjoint to multiplication by h_k with respect to the Hall inner product.

Theorem 5.5 (Generalised Pieri Rule). *Let $k \geq 1$. The following identities hold in Λ_K:*

$$h_k[a_i] G_s = \sum_t \langle U_k \cdot s, t \rangle \, G_t$$

and

$$h_k[a_i] F_s = \sum_t \langle D_k \cdot t, s \rangle \, F_t.$$

The dual identities are:

$$h_k^\perp G_s = \sum_t \langle D_k \cdot s, t \rangle \, G_t$$

and

$$h_k^\perp F_s = \sum_t \langle U_k \cdot t, s \rangle \, F_t.$$

Proof. Follows immediately from the definitions of U_k, D_k and $h_k[a_i]$ together with Theorem 5.4 and the comments immediately after it. ☐

Define a map $\kappa : \Lambda \to K$ by $p_k \mapsto a_k$.

Theorem 5.6 (Generalised Cauchy Identity). *We have the following identity in the completion of $\Lambda_K(X) \otimes \Lambda_K(Y)$:*

$$\sum_s F_s(X) G_s(Y) = \prod_{i,j} \left(1 + \kappa(h_1) x_i y_j + \kappa(h_2)(x_i y_j)^2 + \cdots \right).$$

More generally, let $r, t \in S$. Then we have

$$\sum_s F_{s/t}(X) G_{s/r}(Y) = \prod_{i,j} \left(1 + \kappa(h_1) x_i y_j + \kappa(h_2)(x_i y_j)^2 + \cdots \right) \sum_s F_{r/s}(X) G_{t/s}(Y). \quad (5.4)$$

Proof. We know $[B_k, B_l] = k a_k \delta_{k,-l}$. An argument similar to the proof of Theorem 4.2 gives the identity

$$D_b U_a = \sum_{i=0}^{m} \kappa(h_i) U_{a-i} D_{b-i}$$

where $m = \min(a, b)$. Let $U(x) := 1 + \sum_{i>0} U_i x^i$ and similarly $D(x) := 1 + \sum_{i>0} D_i x^i$. The above identity is equivalent to

$$D(y) U(x) = U(x) D(y) \left(1 + \kappa(h_1) xy + \kappa(h_2)(xy)^2 + \cdots \right).$$

Now notice that by definition we have $F_{s/t} = \langle \cdots U(x_3) U(x_2) U(x_1) \cdot v_t, v_s \rangle$ and $G_{s/t} = \langle \cdots D(x_3) D(x_2) D(x_1) \cdot v_s, v_t \rangle$. The infinite products make sense since in most factors we are picking the term equal to 1. Thus

$$\begin{aligned}
&\sum_s F_{s/t}(X)G_{s/r}(Y)\\
&= \langle \cdots D(y_3)D(y_2)D(y_1)\cdots U(x_3)U(x_2)U(x_1)\cdot v_t, v_r\rangle\\
&= \prod_{i,j\geq 1}^{\infty}\left(1+\kappa(h_1)x_iy_j+\kappa(h_2)(x_iy_j)^2+\cdots\right)\\
&\qquad \langle \cdots U(x_3)U(x_2)U(x_1)\cdots D(y_3)D(y_2)D(y_1)\cdot v_t, v_r\rangle\\
&= \prod_{i,j\geq 1}^{\infty}\left(1+\kappa(h_1)x_iy_j+\kappa(h_2)(x_iy_j)^2+\cdots\right)\sum_s G_{t/s}(Y)F_{r/s}(X).
\end{aligned}$$

These manipulations of infinite generating functions make sense since they are well defined when restricted to a finite subset of the variables $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$. $\square$

The results of this Section are related to results of Fomin [11, 9, 10] and of Bergeron and Sottile [3]. Fomin studies combinatorial operators on posets and recovers Cauchy style identities similar to ours. His approach is more combinatorial and he focuses on generalising Schensted style algorithms to these more general situations. Bergeron and Sottile have also made definitions similar to our $F_{s/t}$. Their interests have been towards aspects related to Hopf algebras and non-commutative symmetric functions; see also [8, 4].

In fact, a converse to Theorem 5.6 exists. Suppose that $\{B'_k : k \in \mathbb{Z}\backslash\{0\}\}$ are operators acting on a vector space V with a distinguished basis $\{v_s : s \in S\}$. Suppose $\{B'_k \mid k > 0\}$ and $\{B'_k \mid k < 0\}$ are both commuting sets of operators, and define U'_k and D'_k as with U_k and D_k. Then we can define $F'_{s/t}(X) := \sum_\alpha x^\alpha \left\langle U'_{\alpha_l}U'_{\alpha_{l-1}}\cdots U'_{\alpha_1}\cdot t, s\right\rangle$ and similarly for $G'_{s/t}$.

Suppose first that the Generalised Cauchy identities (5.4) hold for F' and G' and some coefficients b_i in place of $\kappa(h_i)$. Then by the argument in the proof of Theorem 5.6, we must have

$$\left\langle\left(D'(y)U'(x) - U'(x)D'(y)\left(1+b_1xy+b_2(xy)^2+\cdots\right)\right)\cdot v_t, v_r\right\rangle = 0$$

for every $t, r \in S$. This implies that $D'(y)U'(x) = U'(x)D'(y)\left(1+b_1xy+b_2(xy)^2+\cdots\right)$. Now using the argument in Theorem 4.2 again, we deduce that $[B'_k, B'_l] = ka_k\delta_{k,-l}$ where $a_k = \kappa'(p_k)$ for the map $\kappa' : \Lambda \to K$ given by $\kappa'(h_k) = b_k$.

Now suppose instead of the Generalised Cauchy identity we assume that the Generalised Pieri rules of Theorem 5.5 hold for the family $\{G'_s \in \Lambda_K\}$ and some non-zero parameters a_i. If in addition the family $\{G'_s\}$ is linearly independent in Λ_K then the action of $\{B'_k\}$ on V is isomorphic to the action of $\{h_k[a_i], h_k^\perp\}$ on $\mathrm{span}_K\{G'_s\}$ under the isomorphism $v_s \mapsto G'_s$. Thus the Generalised Pieri rules together with the fact that the family $\{G'_s\}$ is a basis implies that the operators $\{B'_k\}$ generate a copy of the Heisenberg algebra H with parameters a_i.

In conclusion we have the following theorem.

Theorem 5.7. *Suppose $\{B'_k : k \in \mathbb{Z}\backslash\{0\}\}$ are a sequence of operators acting on V, with distinguished basis $\{v_s : s \in S\}$, so that B'_k and B'_l commute if k and l have the same sign. Let $\{a_k \in K\}$ be a sequence of non-zero parameters. Define $F'_{s/t}$, $G'_{s/t}$, U'_k and D'_k as* *before. Suppose in addition that* $\{G'_s \mid s \in S\}$ *are linearly independent. Then the following are equivalent:*

1. *The operators* $\{B'_k\}$ *generate an action of the Heisenberg algebra with parameters* a_i*.*
2. *The family* $\{G'_s\}$ *satisfies the conclusions of Theorem 5.5.*
3. *The families* $\{G'_{s/t}\}$ *and* $\{F'_{s/t}\}$ *satisfy the conclusions of Theorem 5.6.*

5.3 Examples

5.3.1 Schur functions

If $V = \mathcal{F}^{(0)}$ and $H_{\text{Schur}} = H$ acts as in Section 5.1, then Theorem 5.4 is just Theorem 5.2, where the indexing set S can be identified with the set of partitions $\mathcal{P}$. In this case, the operators B_k and B_{-k} are adjoint with respect to $\langle ., . \rangle$ and so $F_\lambda = G_\lambda = s_\lambda$ for every λ. The definition of $s_{\lambda/\mu} = F_{\lambda/\mu}$ in terms of the operators U_k is exactly the usual combinatorial definition of skew Schur functions in terms of semistandard Young tableaux. The symmetric function $h_k[a_i] = h_k$ is the usual homogeneous symmetric function and the coefficients $\langle U_k \cdot \lambda, \mu \rangle$ are equal to 1 if μ/λ is a horizontal strip of size k and equal to 0 otherwise. The coefficients $\kappa(h_i)$ are all equal to 1 and Theorem 5.6 reduces to the usual Cauchy identity.

5.3.2 Direct sums

Let V_1 and V_2 be two representations of H with distinguished bases $\{v_{s_1} : s_1 \in S_1\}$ and $\{v_{s_2} : s_2 \in S_2\}$ respectively. Then $V = V_1 \oplus V_2$ is a representation of H with distinguished basis $\{v_s \mid s \in S_1 \amalg S_2\}$. If $s, t \in S_i$ for some i then $F^V_{s/t} = F^{V_i}_{s/t}$ otherwise if for example $s \in S_1$ and $t \in S_2$ we have $F^V_{s/t} = 0$. Thus the family of symmetric functions that we obtain from V is the union of the families of symmetric functions we obtain from V_1 and V_2.

5.3.3 Tensor products

Let V_1 and V_2 be two representations of H with distinguished bases $\{v_{s_1} : s_1 \in S_1\}$ and $\{v_{s_2} : s_2 \in S_2\}$ respectively, as before. Then $V_1 \otimes V_2$ has a distinguished basis $\{v_{s_1} \otimes v_{s_2} \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}$. Let a Heisenberg algebra $H^{(2)} = \tilde{H}$ act on $V_1 \otimes V_2$ by defining the action of $\tilde{B}_k$ by

$$\tilde{B}_k \cdot v_1 \otimes v_2 = (B_k \cdot v_1) \otimes v_2 + v_1 \cdot (B_k \cdot v_2).$$

This action is natural when one views Λ as a Hopf algebra. If the original Heisenberg algebra H had parameters a_i then one can check that the new Heisenberg algebra $\tilde{H}$ has parameters $\tilde{a}_i = 2a_i$. The action of $\tilde{U}_k$ is given by

$$\tilde{U}_k \cdot v_1 \otimes v_2 = \sum_{i=0}^{k} (U_i \cdot v_1) \otimes (U_{k-i} \cdot v_2)$$

and similarly for $\tilde{D}_k$. By definition, one sees that $F_{s_1 \otimes s_2/t_1 \otimes t_2} = F_{s_1/t_1} F_{s_2/t_2}$ and similarly for the G-functions. Thus the family of symmetric functions we obtain from $V = V_1 \otimes V_2$ are pairwise products of the symmetric functions we obtain from V_1 and V_2.

More generally, the tensor products $V_1 \otimes \cdots \otimes V_n$ leads to generating functions which are products $F_{s_1/t_1} \cdots F_{s_n/t_n}$ of n original generating functions. We will denote the Heisenberg algebra acting on this tensor product by $H^{(n)}$.

5.3.4 Macdonald polynomials

Let $K = \mathbb{Q}(q,t)$ and $P_\lambda(X;q,t)$ and $Q_\lambda(X;q,t)$ be the Macdonald polynomials introduced in [46]. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and $s = (i,j) \in \lambda$ be a square. Then the arm-length of s is given by $a_\lambda(s) = \lambda_i - j$ and the leg-length of s is given by $l_\lambda(s) = \lambda'_j - i$. Now let s be any square. Define ([46, Chapter VI, (6.20)])

$$b_\lambda(s) = b_\lambda(s;q,t) = \begin{cases} \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}} & \text{if } s \in \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Now let λ/μ be a horizontal strip. Let $C_{\lambda/\mu}$ (respectively $R_{\lambda/\mu}$) denote the union of columns (respectively rows) that intersect $\lambda - \mu$. Define ([46, Chapter VI, (6.24)])

$$\phi_{\lambda/\mu} = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}$$

and

$$\psi_{\lambda/\mu} = \prod_{s \in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{b_\mu(s)}{b_\lambda(s)}.$$

Let $V_{\text{Mac}} = \mathbf{F}_K$ be the vector space over K with distinguished basis labelled by partitions. Define operators $\{U_k, D_k : k \in \mathbb{Z}_{>0}\}$ by:

$$U_k \cdot \lambda = \sum_\mu \phi_{\mu/\lambda}\mu, \qquad D_k \cdot \lambda = \sum_\mu \psi_{\lambda/\mu},$$

where the sums are over horizontal strips of size $|k|$. Then $Q_{\lambda/\mu} = F_{\lambda/\mu}$ and $P_{\lambda/\mu} = G_{\lambda/\mu}$, so in particular the operators $\{U_k \mid k \in \mathbb{Z}_{>0}\}$ commute and so do the operators $\{D_k \mid k \in \mathbb{Z}_{>0}\}$. Now we have ([46, Ex.7.6])

$$\sum_\rho Q_{\rho/\lambda}(X;q,t)P_{\rho/\mu}(Y;q,t) = \left(\sum_\sigma Q_{\mu/\sigma}(X;q,t)P_{\lambda/\sigma}(Y;q,t)\right)\prod_{i,j}\prod_{r=0}^{\infty}\frac{1-tx_iy_jq^r}{1-x_iy_jq^r}.$$

The product $\prod_{r=0}^{\infty}\frac{1-ytq^r}{1-yq^r}$ can be written as $\sum_{n\geq 0} g_n(1,0,0,\ldots;q,t)y^n$ where g_n is given by ([46, Chapter VI, (2.9)])

$$g_n(X;q,t) = \sum_{\lambda \vdash n} z_\lambda(q,t)^{-1}p_\lambda(X)$$

where $z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$. Thus $b_n := g_n(1,0,0,\ldots;q,t) = n\sum_{\lambda\vdash n} z_\lambda(q,t)^{-1}$. Using Theorem 5.7, we see that the operators $\{U_k, D_k \mid k \in \mathbb{Z}_{>0}\}$ generate a copy of a Heisenberg algebra H_{Mac}. A short calculation shows that the parameters $a_k \in \mathbb{Q}(q,t)$ are given by $a_k = \frac{1-t^k}{1-q^k}$.

In fact Theorem 5.7 shows that the Pieri (and dual Pieri) rule for Macdonald polynomials is equivalent to the (generalised) Cauchy identity for Macdonald polynomials.

5.3.5 Ribbon functions

The actions of $B_{-k} = p_k(\mathbf{u})$ and $B_k = p_k^{\perp}(\mathbf{u})$ of Chapter 4 are adjoint. Theorem 5.4 is a generalisation of Theorem 4.10. Theorems 5.6 and 5.5 are generalisations of Theorems 4.28 and 4.12.

At $q = 1$, the Fock space $\mathbf{F}$ for $U_q(\widehat{\mathfrak{sl}}_n)$ should be thought of as a sum of tensor products:

$$\mathbf{F} \cong \bigoplus_{n\text{-cores}} (\mathcal{F}^{(0)})^{\otimes n} \tag{5.5}$$

where $\mathcal{F}^{(0)}$ is the Fock space which leads to Schur functions. Combinatorially, the decomposition (5.5) is given by writing a partition in terms of its n-core and its n-quotient; see Section 2.1. As shown in Section 5.3.3, the F functions we obtain in this way are products of n of the F functions for $\mathcal{F}^{(0)}$, that is, (skew) Schur functions. This is simply the formula $\mathcal{G}_\lambda(X; 1) = s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(n-1)}}$ observed in Section 2.2. In fact, the $q = 1$ specialisation corresponds to action of the Heisenberg algebra commuting with the action of $\widehat{\mathfrak{sl}}_n$ on $\mathbf{F}$. In the following sections we will try to generalise ribbon functions to other quantum affine algebras. However, it will not be possible in general to specialise $q = 1$ to obtain a classical Fock space representation since in some cases the parameters a_i may have a pole at $q = 1$.

It would be interesting to see whether ribbon functions and Macdonald polynomials can be combined by finding a deformation of the action of $(H_{\mathrm{Mac}})^{(n)}$ on $V_{\mathrm{Mac}}^{\otimes n}$.

5.4 Ribbon functions of classical type

5.4.1 KMPY

In [27], Kashiwara, Miwa, Petersen and Yung give a general construction of q-deformed Fock spaces using perfect crystals. These Fock spaces are generalisations of the Fock space constructed in [28] which correspond to the quantum affine algebra $A_n^{(1)}$. The Fock spaces constructed in [27] correspond to the level 1 basic representations of the quantum affine algebras of types $A_{2n}^{(2)}$, $B_n^{(2)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. A more combinatorial description of these Fock spaces was given by Kang and Kwon [24] in terms of objects known as *Young walls*. However, the description of the action of the Heisenberg algebra is not available in terms of Young walls so we will only use the description in [27].

A huge amount of notation and machinery is developed for the construction in [27], so we will not be able to give all the details of the theory but will emphasize the parts important for our work.

Let $\mathfrak{g}$ be an affine Lie algebra. Let $\{\alpha_i \in \mathfrak{h}^*\}$ for $i \in I$ denote the set of simple roots and let $\{h_i \in \mathfrak{h}\}$ denote the set of simple coroots. Let $P \subset \mathfrak{h}^*$ denote the weight lattice and let $Q \subset \mathfrak{h}^*$ denote the root lattice. Let $\delta \in Q$ be an element such that $\mathbb{Z}\delta = \{\lambda \in Q \mid \langle h_i, \lambda\rangle = 0\}$ where $\langle .,. \rangle$ is the coupling $\mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$. Let $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$ denote the classical part of P. Let $c \in \sum_i \mathbb{Z}_{>0} h_i$ be an element such that $\mathbb{Z}c = \{h \in \sum_i \mathbb{Z}h_i \mid \langle h, \alpha_i\rangle = 0\}$. The *level* of $\lambda \in P$ is given by $\langle c, \lambda\rangle$. Let P^0 and P^0_{cl} denote the level 0 parts of the weight lattice and classical weight lattice. Let W denote the Weyl group of $\mathfrak{g}$.

Let $U_q(\mathfrak{g})$ denote the quantized enveloping algebra with $\{q^h \mid h \in P\}$ as its Cartan part. Let $U'_q(\mathfrak{g})$ denote the quantized enveloping algebra with $\{q^h \mid h \in P_{\mathrm{cl}}\}$ as its Cartan part. Thus $U'_q(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{g})$ is generated over $\mathbb{Q}(q)$ by the generators $\{q^h \mid h \in P\}$ and e_i, f_i for $i \in I$. We will not write down the relations here, but they are nearly identical to the description for type $A_n^{(1)}$ in Section 4.1.2. Also the choice of a coproduct for $U_q(\mathfrak{g})$ will affect all the definitions in the following sections, but we have decided to hide this dependence; see [27, Section 2.2] for details.

We briefly sketch the idea behind the construction. We begin with a representation V_{aff} of $U_q(\mathfrak{g})$. The aim is to define a representation "$\wedge^\infty V_{\mathrm{aff}}$". To begin with one wants to define $\wedge^2 V_{\mathrm{aff}}$ by taking $V_{\mathrm{aff}}^{\otimes 2}$ and quotienting by relations of the form $v_1 \otimes v_2 = -v_2 \otimes v_1$. This definition fails since $U_q(\mathfrak{g})$ is not a cocommutative Hopf algebra. So instead one defines the subspace

$$N = U_q(\mathfrak{g})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z](u \otimes u)$$

for an extremal vector $u \in V_{\mathrm{aff}}$. We then let $\wedge^2 V_{\mathrm{aff}} := V_{\mathrm{aff}}^{\otimes 2}/N$ and define $\wedge^\infty V_{\mathrm{aff}}$ similarly. However, even given a nice basis of V_{aff} it is not clear how to write down a basis for $\wedge^\infty V_{\mathrm{aff}}$. This is where the use of perfect crystals, global crystal bases and energy functions will be essential.

5.4.2 Crystal bases, perfect crystals and energy functions

Crystals

Crystal bases were first introduced by Kashiwara [25]. Let A denote the set of all functions $f \in \mathbb{Q}(q)$ which are regular at $q = 0$. Let $\tilde{e}_i, \tilde{f}_i$ denote the *Kashiwara operators* introduced in [25]. A *crystal lattice* L is a free A-submodule of V, compatible with the weight decomposition of V, such that $V = \mathbb{Q}(q) \otimes_A L$ and such that $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for each $i \in I$. A *crystal basis* (L, B) of V consists of a crystal lattice L of V and a $\mathbb{Q}$-basis B of L/qL. The set B is compatible with the weight decomposition, satisfies $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$ for each $i \in I$ and satisfies $\tilde{f}_i(b) = b' \iff \tilde{e}_i(b') = b$ for any $b, b' \in B$. The main theorem concerning crystal bases is that a unique (up to isomorphism) crystal basis exists for the highest weight irreducible representation V_λ of $U_q(\mathfrak{g})$.

The set B, together with its additional structure is called a *crystal*. It can be thought of as a directed graph with edges labelled by the index set I. One can define crystals abstractly as follows.

A $U_q(\mathfrak{g})$-crystal is a nonempty set B together with maps $\mathrm{wt} : B \to P$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ for all $i \in I$ and $\epsilon_i : B \to \mathbb{Z}$ and $\phi_i : B \to \mathbb{Z}$ satisfying:

$$\begin{aligned}
\tilde{f}_i(b) = b' &\iff \tilde{e}_i(b') = b && \text{for } b, b' \in B,\\
\mathrm{wt}(\tilde{f}_i(b)) &= \mathrm{wt}(b) - \alpha_i && \text{if } \tilde{f}_i(b) \in B,\\
\langle h_i, \mathrm{wt}(b)\rangle &= \phi(b) - \epsilon_i(b) && \text{for } i \in I \text{ and } b \in B,\\
\epsilon_i(b) &= \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq \emptyset\} && \text{for } i \in I \text{ and } b \in B,\\
\phi_i(b) &= \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq \emptyset\} && \text{for } i \in I \text{ and } b \in B.
\end{aligned}$$

The most important combinatorial operation for a crystal is the tensor product operation, which corresponds exactly to taking tensor products of the associated representations.

Perfect crystals

Perfect crystals were first introduced in [22, 23] to compute one-point functions of the vertex models in 2-dimensional lattice statistical models. Let B be the crystal corresponding to a crystal base (L, B) of an integrable finite-dimensional $U_q'(\mathfrak{g})$-module of level 0. The crystal B is *perfect* of level $l > 0$ if:

1. The crystal $B \otimes B$ is connected;

2. There exists $\lambda \in P^0_{\mathrm{cl}}$ such that $\#(B_\lambda) = 1$ and all weights of V are contained in the convex hull of $W \cdot \lambda$.

3. Let $\epsilon(b) := \sum_{i \in I} \epsilon_i(b)\Lambda^{\mathrm{cl}}_i \in P_{\mathrm{cl}}$ and $\phi(b) := \sum_{i \in I} \phi_i(b)\Lambda^{\mathrm{cl}}_i \in P_{\mathrm{cl}}$, where Λ^{cl}_i is the classical part of Λ_i. Then for any $b \in B$, we have $\langle c, \epsilon(b)\rangle \geq l$.

4. The maps ϵ and ϕ from B_{min} to $(P^+_{\mathrm{cl}})_l$ are bijective. Here $B_{\mathrm{min}} := \{b \in B \mid \langle c, \epsilon(b)\rangle = l\}$ and $(P^+_{\mathrm{cl}})_l := \{\lambda \in P_{\mathrm{cl}} \mid \langle c, \lambda\rangle = l$ and $\langle h_i, \lambda\rangle \geq 0$ for every $i \in I\}$.

Energy function

Let V be an integrable finite-dimensional representation of $U'_q(\mathfrak{g})$. One can define a representation V_{aff} of $U_q(\mathfrak{g})$, which may be identified with $V \otimes \mathbb{C}[z, z^{-1}]$ (where z has "weight" δ). Assume that

> **(P)**: V has a perfect crystal base (L, B).

If (L, B) is a crystal base for V, then we have a crystal base $(L_{\mathrm{aff}}, B_{\mathrm{aff}})$ for V_{aff}. The elements of B_{aff} are labelled $z^n b$ where $n \in \mathbb{Z}$ and $b \in B$. The level l of the perfect crystal B will determine the level of the Fock space we end up constructing. However, some of the assumptions in the following were only verified for certain crystals, case by case, in [27].

Also assume that

> **(G)**: V has a lower global base $\{G(b)\}_{b \in B}$.

A lower global base ([26]), see also 4.1.3, is a "lifting" of the crystal basis to V. The lower global base of V_{aff} satisfies $G(z^n b) = z^n G(b)$ for $n \in \mathbb{Z}$ and $b \in B_{\mathrm{aff}}$.

An *energy function* $H : B_{\mathrm{aff}} \otimes B_{\mathrm{aff}} \to \mathbb{Z}$ satisfies:

1. $H(zb_1 \otimes b_2) = H(b_1 \otimes b_2) - 1$.

2. $H(b_1 \otimes zb_2) = H(b_1 \otimes b_2) + 1$.

3. H is constant on every component of the crystal graph $B_{\mathrm{aff}} \otimes B_{\mathrm{aff}}$.

The above conditions determine H up to a constant. We normalise H by requiring that $H(b \otimes b) = 0$ for any element $b \in B_{\mathrm{aff}}$ of extremal weight.

The existence of an energy function is shown using the R-matrix; see [22]. The R-matrix is a $U_q(\mathfrak{g})$-linear map from $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$ to its completion $V_{\mathrm{aff}} \hat{\otimes} V_{\mathrm{aff}}$ such that

$$R \circ (z \otimes 1) = (1 \otimes z) \circ R$$
$$R \circ (1 \otimes z) = (z \otimes 1) \circ R.$$

The energy function and the R-matrix are related by

$$R(G(b_1) \otimes G(b_2)) \equiv G(z^{H(b_1 \otimes b_2)} b_1) \otimes G(z^{-H(b_1 \otimes b_2)} b_2) \mod qL_{\mathrm{aff}} \hat{\otimes} L_{\mathrm{aff}}$$

for every $b_1, b_2 \in B_{\mathrm{aff}}$. It is known that R has finitely many poles. This means that there is a $\psi \in \mathbb{Q}(q)[z \otimes z^{-1}, z^{-1} \otimes z]$ such that ψR sends $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$ into itself. Assume that

> **(D)**: $\psi \in A[z \otimes z^{-1}]$ and $\psi = 1$ at $q = 0$.

Let $s : P \to \mathbb{Q}$ be a linear form such that $s(\alpha_i) = 1$ for all $i \in I$. Let $l : B_{\mathrm{aff}} \to \mathbb{Z}$ be defined by $l(b) = s(\mathrm{wt}(b)) + c$, where c is chosen so that l is $\mathbb{Z}$-valued. The construction of [27] depends on the assumption that

> **(L)**: If $H(b_1 \otimes b_2) \leq 0$ then $l(b_1) \geq l(b_2)$.

Let $u \in V_{\mathrm{aff}}$ be an extremal vector and let

$$N = U_q(\mathfrak{g})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z](u \otimes u).$$

In [27] one also assumes that

> **(R)**: For every $b_1, b_2 \in B_{\mathrm{aff}}$ such that $H(b_1 \otimes b_2) = 0$, we have $C_{b_1,b_2} \in N$ of the form
> $$C_{b_1,b_2} = G(b_1) \otimes G(b_2) - \sum_{b_1',b_2'} a_{b_1',b_2'} G(b_1') \otimes G(b_2')$$
> where $a_{b_1',b_2'} \in \mathbb{Z}[q, q^{-1}]$ and the sum is over all b_1' and b_2' such that $H(b_1' \otimes b_2') > 0$ and $l(b_2) \leq l(b_1') < l(b_1)$ and $l(b_2) < l(b_2') \leq l(b_1)$.

It is conjectured that the existence of the perfect crystal **(P)** and lower global base **(G)** is sufficient to imply the other conditions **(D)**, **(L)** and **(R)**.

5.4.3 The Fock space and the action of the bosons

Under the assumptions **(P)**, **(G)**, **(D)**, **(L)** and **(R)** Kashiwara, Miwa, Petersen and Yung define a Fock space $\mathcal{F}_m$. These assumptions are verified for level 1 perfect crystals of types $A_n^{(1)}$, $A_{2n}^{(2)}$, $B_n^{(2)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. One first defines

$$\wedge^2 V_{\mathrm{aff}} = V_{\mathrm{aff}}^{\otimes 2}/N$$

and shows that $\wedge^2 V_{\mathrm{aff}}$ is spanned by $\{G(b_1) \otimes G(b_2) \mid H(b_1 \otimes b_2) > 0\}$ which are called normally ordered. Informally, the relations **(R)** allows an arbitrary tensor to be written as a sum of normally ordered tensors.

The Fock space $\mathcal{F}_m$ is then a suitable inductive limit as $k \to \infty$ of $\wedge^k V_{\mathrm{aff}}$. Suppose our perfect crystal B has level l. A sequence $\{b_m^\circ\}_{m\in\mathbb{Z}}$ in B_{aff} is called a ground state sequence if

$$\langle c, \epsilon(b_m^\circ)\rangle = l; \quad \epsilon(b_m^\circ) = \phi(b_{m+1}^\circ); \quad H(b_m^\circ \otimes b_{m+1}^\circ) = 1.$$

Define weights $\lambda_m \in P$ of level l by $\lambda_m = \mathrm{wt}(b_m^\circ) + \lambda_{m+1}$ and $\mathrm{cl}(\lambda_m) = \phi(b_m^\circ) = \epsilon(b_{m-1}^\circ)$ where $\mathrm{cl} : P \to P_{\mathrm{cl}}$. Set $v_m^\circ := G(b_m^\circ)$.

A sequence $(b_m, b_{m+1}, \ldots)$ in B_{aff} such that $b_k = b_k^\circ$ for sufficiently large k is called *normally ordered* if $H(b_i \otimes b_{i+1}) > 0$ for all $i \geq m$. The main property of the Fock space $\mathcal{F}_m$ is that the *normally ordered wedges* $G(b_m) \wedge G(b_{m+1}) \wedge \cdots$ form a base of $\mathcal{F}_m$. In [27] the action of the corresponding quantized affine algebra $U_q(\mathfrak{g})$ on $\mathcal{F}_m$ is given. We will not explain the details here, trusting that Section 4.1.2 gives a flavour of what one might expect.

We should remark that the Fock space $\mathcal{F}_m$ is endowed with a *q-adic topology.* All formulae should be shown to be convergent in this topology since a priori some of the formulae below may involve an infinite number of terms. We will however, ignore this technicality in the subsequent discussion.

Let $u_m, u_{m+1}, \ldots \in V_{\mathrm{aff}}$ satisfy $u_k = v_k^\circ$ for k sufficiently large. Define the action of B_n for $n \neq 0$ by

$$B_n \cdot u_m \wedge u_{m+1} \wedge \cdots = (z^n u_m \wedge u_{m+1} \wedge u_{m+2} \wedge \cdots) + (u_m \wedge z^n u_{m+1} \wedge u_{m+2} \wedge \cdots) + \cdots . \quad (5.6)$$

Theorem 5.8 ([27]). *The definition above extends to an action of a Heisenberg algebra H on $\mathcal{F}_m$. As a $U_q'(\mathfrak{g}) \otimes H$ module, we have*

$$\mathcal{F}_m \cong V_{\lambda_m} \otimes \mathbb{Q}[H_-],$$

where V_{λ_m} is the irreducible $U_q'(\mathfrak{g})$-module with highest weight λ_m.

The parameters a_i of the Heisenberg algebra $H = H^\Phi$ are calculated case by case in [27].

5.4.4 Ribbons, ribbon strips and ribbon functions

Let Φ be one of the affine Dynkin types $A_n^{(1)}$, $A_{2n}^{(2)}$, $B_n^{(2)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. Fix a ground state sequence $(b_m^\circ, b_{m+1}^\circ, \ldots)$ and let $\mathcal{F} := \mathcal{F}_m$ denote the q-deformed Fock space described above.

Let S denote the set of normally ordered sequences $\{(b_m, b_{m+1}, \ldots)\}$. For an element $s = (b_m, b_{m+1}, \ldots) \in S$ let $G(s)$ denote the normally ordered wedge $G(b_m) \wedge G(b_{m+1}) \wedge \cdots$. As usual, define an inner product $\langle ., . \rangle$ on $\mathcal{F}$ by requiring that the normally ordered wedges form an orthonormal basis. Give S and thus $\mathcal{F}$ a $\mathbb{Z}$-grading by defining

$$\deg(b_m, b_{m+1}, \ldots) = \sum_{i=m}^{\infty} l(b_i^\circ) - l(b_i).$$

It is clear from [27] that $G(b_m^\circ) \wedge G(b_m^\circ) \wedge \cdots$ is a highest weight vector for the action of H on $\mathcal{F}$. Let $s^\circ = (b_m^\circ, b_{m+1}^\circ, \ldots)$.

Proposition 5.9. *The action of H on $\mathcal{F}$ is $\mathbb{Z}$-graded.*

Proof. The function $l : B_{\mathrm{aff}} \to \mathbb{Z}$ satisfies $l(zb) = l(b) + a$ for some positive integer a not depending on b; see [27, p.14]. The subspace $N \subset V_{\mathrm{aff}}^{\otimes 2}$ is a homogeneous subspace with grading given by $\deg(G(b) \otimes G(b')) = l(b) + l(b')$. Thus the normal ordering relations are compatible with the $\mathbb{Z}$-grading of $\mathcal{F}$. So the action of B_k has degree $-ka$. □

A *ribbon of type* Φ is a pair $(b, b') \in S$ (also written b'/b) such that

$$\langle U_1 \cdot G(b), G(b') \rangle \neq 0.$$

More generally, a *horizontal ribbon strip of type* Φ *and size* k is a pair $(b, b') \in S$ (also written b'/b) such that $\langle U_k \cdot G(b), G(b') \rangle \neq 0$. It is not always the case that the coefficients $\langle U_k \cdot G(b), G(b') \rangle$ are pure powers of q but it seems reasonable to formally define $q^{\mathrm{spin}(b'/b)} := \langle U_k \cdot G(b), G(b') \rangle$. By Proposition 5.9, a horizontal ribbon strip b'/b has a well-defined size.

Define the symmetric functions $F_{s/t}, G_{s/t} \in \Lambda_K = \Lambda(q)$ for each $s, t \in S$ as in Section 5.2.1, which we call *(skew) ribbon functions of type* Φ. We have $F_s := F_{s/s^\circ}$ and $G_s := G_{s/s^\circ}$. I do not know whether the symmetric functions F and G are the same or not. This is the case for $\Phi = A_n^{(1)}$ and for $\Phi = A_{2n}^{(2)}$ it would be a consequence of the conjecture on p.74 of [27]. By Theorems 5.4, 5.5 and 5.6 we know that the symmetric functions F_s and

G_s satisfy Cauchy and Pieri rules and are also images of a Boson-Fermion correspondence. This suggests that ribbon functions of type Φ should be of independent interest.

In particular, what can one say about the q-Littlewood Richardson coefficients $c^{\lambda}_{s/s'}(q) \in K$ of type Φ? They are given by either

$$F_{s/s'}(X) = \sum_{\lambda} c^{\lambda}_{s/s'}(q) s_{\lambda}(X)$$

or

$$S_{\lambda} \cdot G(s') = \sum_{s} c^{\lambda}_{s/s'}(q) G(s).$$

We speculate that these coefficients $c^{\lambda}_{s/s'}(q)$ can be expressed as power series in q with integer coefficients. We shall return to the element $S_{\lambda} \cdot G(s')$ later. When $s' = s^{\circ}$, we set $c^{\lambda}_{s}(q) := c^{\lambda}_{s/s^{\circ}}(q)$.

For the affine Dynkin types $\Phi \in \{A^{(2)}_{2n}, B^{(1)}_{n}, A^{(2)}_{2n-1}, D^{(1)}_{n}\}$, the parameters $\{a_i\}$ for the Heisenberg algebra H^{Φ} acting on $\mathcal{F}$ are given by

$$a_i = \frac{1+\xi^i}{1-p^{2i}}$$

where ξ and p are given below.

Φ	$A^{(2)}_{2n}$	$B^{(1)}_{n}$	$A^{(2)}_{2n-1}$	$D^{(1)}_{n}$
p	q^2	q^2	q	q
ξ	$-q^{2(2n+1)}$	$q^{2(2n-1)}$	$-q^{2n}$	q^{2n-2}

The Cauchy identity of Theorem 5.6 thus takes the form

$$\sum_{s} F_s(X) G_s(Y) = \prod_{i,j=1}^{\infty} \prod_{k=0}^{\infty} \frac{1}{(1-p^{2k}x_i y_j)(1-\xi p^{2k} x_i y_j)}.$$

For $\Phi = D^{(2)}_{n+1}$, the parameters $\{a_i\}$ are given by

$$a_i = \begin{cases} \frac{1+q^{4ni}}{1-2q^{2i}-q^{4ni}} & \text{for } m \in 2\mathbb{Z}, \\ 1 & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

5.4.5 The case $\Phi = A^{(1)}_{n-1}$

When $\Phi = A^{(1)}_{n-1}$ we recover the theory of n-ribbon functions studied in Chapter 4. We take B to be the perfect crystal with n elements $\{b_j\}_{j \in [0,n-1]}$. For our purposes, the crystal structure will not be important. The affine crystal B_{aff} has elements

$$\{b_{nk+j'} := z^k b_{j'} \mid k \in \mathbb{Z} \text{ and } j' \in [0, n-1]\}.$$

Set $v_j := G(b_j)$. The energy function $H : B_{\mathrm{aff}} \otimes B_{\mathrm{aff}} \to \mathbb{Z}$ is given on $B \otimes B$ by

$$H(b_i \otimes b_j) = \begin{cases} 1 & \text{for } i > j, \\ 0 & \text{for } i \leq j. \end{cases}$$

We shall take the ground state sequence to be $(b_0, b_1, b_2, \ldots)$. There is a bijection between normally ordered sequences $(b_{i_0}, b_{i_1}, \ldots)$ and partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ by setting $\lambda_j = j - 1 - i_{j-1}$. The notation here differs somewhat from Section 5.1. The definition of ribbon tableaux and ribbon functions can be recovered from Equation (5.6) and the normal ordering relations $\{\tilde{C}_{i,j}\} \subset N$ where:

$$\begin{aligned} \tilde{C}_{i,i} &= v_i \otimes v_i && \text{for } i \in [0, n-1], \\ \tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j + q z^{-H(i,j)} v_j \otimes v_i && \text{for } (i,j) \in [0, n-1]^2 \backslash \{(k,k)\}. \end{aligned}$$

Here $H(i,j) := H(b_i \otimes b_j)$. We have followed the notation in [27] and it differs from that in the earlier chapters by the change of variables $q \mapsto -q$. Using the relations $K[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot z^m \otimes z^m \cdot \tilde{C}_{i,j}$ one can reorder any wedge into a normally ordered one.

For example, the n ribbons which can be added to the empty partition correspond to the n terms in

$$B_1 \cdot (v_0 \wedge v_1 \wedge \cdots) = (z^{-1} v_0 \wedge v_1 \wedge \cdots) + (v_0 \wedge z^{-1} v_1 \wedge \cdots) + \cdots + (v_0 \wedge \cdots \wedge z^{-1} v_{n-1} \wedge \cdots).$$

The other places where one may multiply by z^{-1} all vanish.

5.4.6 The case $\Phi = A_{2n}^{(2)}$

Similar analysis may be applicable to the case $\Phi = D_{n+1}^{(2)}$.

Preliminaries

We will work through this example following the notation of Section 5.3 in [27]. Begin with a $(2n+1)$-dimensional $U_q'(A_{2n}^{(2)})$-module V with level 1 perfect crystal $B := \{b_i\}_{i \in [-n,n]}$. Set $v_i := G(b_i) \in V$. Let $l : B_{\text{aff}} \to \mathbb{Z}$ be given by

$$l(z^m b_j) = \begin{cases} (2n+1)m + n + 1 - j & \text{for } j \in [1, n], \\ (2n+1)m & \text{for } j = 0, \\ (2n+1)m - (n+1+j) & \text{for } j \in [-n, -1]. \end{cases}$$

The map l gives a total ordering of B_{aff}. Define the ordering $\succ$ on $[-n, n]$ by

$$1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1 - n \succ \cdots \succ -1.$$

Define the energy functions H on $B \otimes B$ by

$$H(i,j) := H(b_i \otimes b_j) = \begin{cases} 1 & \text{if } i \prec j \text{ or } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $H(0,0) = 1$, the ground state sequence can be taken to be $v_0 \wedge v_0 \wedge \cdots$. There is an injection from normally ordered wedges $G(b_0) \wedge G(b_1) \wedge \cdots$ to the set of partitions given by

$$G(b_0) \wedge G(b_1) \wedge \cdots \longmapsto (-l(b_0), -l(b_1), \ldots).$$

The image of this injection is the set of partitions whose parts are only allowed to repeat if they are multiples of $(2n+1)$.

Normal order relations

The following normal order relations $\tilde{C}_{i,j} \in N$ can be used to reorder wedges to make them normally ordered. Recall that $[2] = q + q^{-1}$ was defined in Section 4.1.2. In the following we assume $i \in [-n, n]$.

$$\begin{aligned}
\tilde{C}_{i,i} &:= v_i \otimes v_i && \text{for } i \neq 0, \\
\tilde{C}_{i,-i} &:= v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} \\
&\quad + q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} + q^4 z^{-H(i,-i)} v_{-i} \otimes v_i && \text{for } i \notin \{-1, 0, n\}, \\
\tilde{C}_{i,j} &:= v_i \otimes z^{-H(i,j)} v_j + q^2 z^{-H(i,j)} v_j \otimes v_i && \text{for } i \neq \pm j, \\
\tilde{C}_{0,0} &:= v_0 \otimes z^{-1} v_0 + q^2 [2] v_{-n} \otimes z^{-1} v_n \\
&\quad + q^2 [2] z^{-1} v_n \otimes v_{-n} + q^2 z^{-1} v_0 \otimes v_0 \\
\tilde{C}_{n,-n} &:= v_n \otimes v_{-n} + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &:= v_{-1} \otimes z^{-1} v_1 + q^4 z^{-1} v_1 \otimes v_{-1}.
\end{aligned}$$

Using the relations $K[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot z^m \otimes z^m \cdot \tilde{C}_{i,j}$ one can reorder any wedge into a normally ordered one.

Let us rewrite these relations in terms of integer sequences and partitions. First identify $G(b_0) \wedge G(b_1) \wedge \cdots$ with $|-l(b_0), -l(b_1), \ldots\rangle$ even if $(b_0, b_1, \ldots)$ is not normally ordered. Then we have the following relations in $\mathcal{F}$, where $i < j$ always lie in $[-n, n]$ and $r = hm$ is an arbitrary non-negative multiple of $h := (2n + 1)$.

$$\begin{aligned}
|\ldots, k, k, \ldots\rangle &= 0 && \text{if } k \neq hm, \\
|\ldots, r-i, r+i, \ldots\rangle &= -q^2 \,|\ldots, r-i+1, r+i-1, \ldots\rangle \\
&\quad - q^2 \,|\ldots, r+i-1, r-i-1, \ldots\rangle \\
&\quad - q^4 \,|\ldots, r+i, r-i, \ldots\rangle && \text{for } i > 1, \\
|\ldots, r+i, r+h-i, \ldots\rangle &= -q^2 \,|\ldots, r+i+1, r+h-i-1, \ldots\rangle \\
&\quad - q^2 \,|\ldots, r+i+1, r+h-i-1, \ldots\rangle \\
&\quad - q^4 \,|\ldots, r+i, r+h-i, \ldots\rangle && \text{for } i > 1, \\
|\ldots, r+i, r+j, \ldots\rangle &= -q^2 \,|\ldots, r+j, r+i, \ldots\rangle && \text{if } i \neq \pm j, \\
|\ldots, r+j, r+h+i, \ldots\rangle &= -q^2 \,|\ldots, r+h+i, r+j, \ldots\rangle && \text{if } i \neq \pm j, \\
|\ldots, r, r+h, \ldots\rangle &= -q^2 [2] \,|\ldots, r+1, r+h-1, \ldots\rangle \\
&\quad - q^2 [2] \,|\ldots, r+h-1, r+1, \ldots\rangle \\
&\quad - q^2 \,|\ldots, r+h, r, \ldots\rangle \\
|\ldots, r-1, r+1, \ldots\rangle &= -q \,|\ldots, r, r, \ldots\rangle - q^4 \,|\ldots, r+1, r-1, \ldots\rangle \\
|\ldots, r+n, r+h-n, \ldots\rangle &= -q^4 \,|\ldots, r+h-n, r+n, \ldots\rangle .
\end{aligned}$$

These relations only allow one to swap entries which do not differ by too much. For entries further apart, we have to apply the element $(1 \otimes z^c + z^c \otimes 1)$ to $\tilde{C}_{i,j}$. In other words we can replace each term $|\ldots, a, b, \ldots\rangle$ by $|\ldots, a + r, b, \ldots\rangle + |\ldots, a, b + r, \ldots\rangle$ for some $r = hm$ in the relations above.

Ribbons for $n = 1$

Let $n = 1$ so that $h = 3$. We calculate

$$B_{-1} \cdot |0, 0, \ldots\rangle = |3, 0, 0, \ldots\rangle + |0, 3, 0, \ldots\rangle + \cdots .$$

The term $|3, 0, \ldots\rangle$ is already a partition. Using the normal order relations, we also have

$$\begin{aligned}
|\ldots, 0, 3, \ldots\rangle &= -q^2[2]\,|\ldots, 2, 1, \ldots\rangle - q^2[2]\,|\ldots, 1, 2, \ldots\rangle - q^2\,|\ldots, 3, 0, \ldots\rangle \\
&= (q^6 - q^2)[2]\,|\ldots, 2, 1, \ldots\rangle - q^2\,|\ldots, 3, 0, \ldots\rangle . \\
|\ldots, 0, 2, 1, \ldots\rangle &= -q^2\,|\ldots, 2, 0, 1, \ldots\rangle = q^4\,|\ldots, 2, 1, 0, \ldots\rangle .
\end{aligned}$$

So,

$$B_{-1} \cdot |0, 0, \ldots\rangle = \frac{1}{1+q^2}\,|3, 0, \ldots\rangle + \left(\sum_{k=1}^{\infty} (-q^2)^k (1 - q^4) \frac{1 + (-1)^{k+1} q^{2k}}{1+q^2} \right) [2]\,|2, 1, 0, \ldots\rangle .$$

Setting $t = -q^2$, we compute that

$$\begin{aligned}
\sum_{k=1}^{\infty} (-q^2)^k (1 + (-1)^{k+1} q^{2k})(1 - q^2) &= \sum_{k=1}^{\infty} t^k (1 - t^k)(1 + t) \\
&= \left(\frac{t}{1-t} - \frac{t^2}{1-t^2} \right) (1 + t) \\
&= \frac{t}{1-t}.
\end{aligned}$$

Thus

$$B_{-1} \cdot |0, 0, \ldots\rangle = \frac{1}{1+q^2}\,|3, 0, \ldots\rangle + \frac{-q^2[2]}{1+q^2}\,|2, 1, 0, \ldots\rangle .$$

So for example, there are the two ribbons (3) and $(2, 1)$ which have q^{spin} equal to $1/(1+q^2)$ and $-q^2[2]/(1+q^2)$ respectively. The ribbon function $F_{(2,1)}(X)$ is equal to $-q^2[2]/(1+q^2)s_1(X)$.

One would like to speculate that the spins of arbitrary ribbons also have a factor of $(1 + q^2)$ in the denominator. At the moment, these brute force computations are not particularly illuminating, but they may become more so if one describes the action of the Bosonic operators in terms of the Young walls in [24]. Also the computation above shows that one cannot naively specialise $q = 1$ in the computation of the action of the Bosonic operators.

5.4.7 Global bases

Imitating Section 4.1.3, we give a possible definition of a global basis for any of the q-deformed Fock spaces $\mathcal{F} := \mathcal{F}_m$ described in [27]. Global bases for the subspace V_{λ_m} were studied in [24]. Our main result will be the analogue of Proposition 4.1.3, connecting these global bases with our generalised ribbon functions. As before we fix a ground state sequence $s^\circ = (b^\circ_m, b^\circ_{m+1}, \ldots)$ and let S denote the set of normally ordered sequences. For $s \in S$ let $G(s)$ denote the corresponding normally ordered wedge.

Define a semi-linear involution $v \mapsto \overline{v}$ on $\mathcal{F}$ by requiring that for any $v \in \mathcal{F}$

$$\begin{aligned} \overline{qv} &= q^{-1}\overline{v}, & & \\ \overline{f_i \cdot v} &= f_i \cdot \overline{v} & & \text{for each } i \in I, \\ \overline{B_k \cdot v} &= B_k \cdot \overline{v} & & \text{for } k < 0, \end{aligned}$$

and requiring that the vacuum vector satisfies $\overline{G(s^\circ)} = G(s^\circ)$. The fact that such an involution exists and is unique follows from Theorem 5.8, and we shall call it the bar involution. When we restrict the bar involution to $V_{\lambda_m} = U'_q(\mathfrak{g}) \cdot G(s^\circ)$ we recover Kashiwara's involution [25] from which global crystal bases are usually defined.

An affirmative answer to the following question would justify the study of this involution. Let $L(\mathcal{F})$ denote the free A-module with basis given by the normally ordered wedges $\{G(s) \mid s \in S\}$; see [27, Section 4].

Question 5.10. *Does there exist a (unique) basis $\{\mathbb{G}_s \mid s \in S\}$ satisfying*

(i) $\overline{\mathbb{G}_s} = \mathbb{G}_s$,

(ii) $\mathbb{G}_s \equiv G(s) \mod qL(\mathcal{F})$*?*

We call such a basis the *global basis* of $\mathcal{F}$ (strictly speaking, it should be called the lower global basis). We now give our analogue of Theorem 4.7 which is due to Leclerc and Thibon. Let λ be a partition. Define $s(\lambda) \in S$ by $s(\lambda) := (z^{-\lambda_1} b^\circ_m, z^{-\lambda_2} b^\circ_{m+1}, \ldots)$.

Theorem 5.11. *Let λ be a partition. Then*

$$S_\lambda \cdot G(s^\circ) \equiv G(s(\lambda)) \mod qL(\mathcal{F}).$$

By definition $S_\lambda \cdot G(s^\circ)$ is also bar-invariant, so that assuming the existence of the global basis, we have $\mathbb{G}_{s(\lambda)} = S_\lambda \cdot G(s^\circ)$. In this case our generalised q-Littlewood Richardson coefficients would again given by coefficients of the global basis: $c^\lambda_s(q) = \langle \mathbb{G}_{s(\lambda)}, G(s) \rangle$.

Proof. The operator S_λ is a $\mathbb{Q}$-linear combination of the Bosonic operators $B_{-\alpha}$. The action of the B_{-k} on a wedge does not involve any powers of q except when applying the normal ordering relations. By Lemma 3.3.2 in [27], the coefficients $a_{b'_1,b'_2}$ of condition (**R**) lie in $q\mathbb{Z}[q]$. Thus reordering wedges can only involve positive powers of q and so in particular $S_\lambda \cdot G(s^\circ) \in L(\mathcal{F})$. To calculate $S_\lambda \cdot G(s^\circ) \mod qL(\mathcal{F})$ we can set $q = 0$.

Then by condition (**R**) the only relations that we need to reorder wedges are of the form

$$\mathbb{Z}[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot (G(b) \otimes G(b')) \in (N \mod qL(\mathcal{F}))$$

for b, b' satisfying $H(b \otimes b') = 0$. Thus for example modulo $N + qL(\mathcal{F})$ we have

$$(z^{-1} \otimes 1) \cdot G(z^a b^\circ_i) \otimes G(z^{a-1} b^\circ_{i+1}) \equiv (-1 \otimes z^{-1}) \cdot G(z^a b^\circ_i) \otimes G(z^{a-1} b^\circ_{i+1}) \tag{5.7}$$

$$G(z^{a-1} b^\circ_i) \otimes G(z^{a-1} b^\circ_{i+1}) \equiv -G(z^a b^\circ_i) \otimes G(z^{a-2} b^\circ_{i+1}). \tag{5.8}$$

The only wedges that occur in $S_\lambda \cdot G(s^\circ)$ at $q = 0$ are of the form

$$G(z^{a_m} b^\circ_m) \wedge G(z^{a_{m+1}} b^\circ_{m+1}) \wedge \cdots$$

for some parameters $a_i \in \mathbb{Z}$. Using $H(b_i^\circ \otimes b_{i+1}^\circ) = 1$, and repeatedly applying $(1 \otimes z^{-1} + z^{-1} \otimes 1)$ to the relation (5.8) one can prove by induction that we have

$$(\cdots G(z^{a_i} b_i^\circ) \wedge G(z^{a_{i+1}} b_{i+1}^\circ) \wedge \cdots) \equiv -(\cdots G(z^{a_{i+1}+1} b_i) \wedge G(z^{a_i - 1} b_{i+1}) \wedge \cdots) \mod qL(\mathcal{F}), \tag{5.9}$$

whenever $a_{i+1} \leq a_i - 2$.

The reordering relation (5.9) is identical to that of Schur functions s_α indexed by a non-negative integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Here one defines s_α by (see [55])

$$s_\alpha(X) = \frac{\det(x_i^{\alpha_j + j - 1})_{i,j=1}^n}{\det(x_i^{j-1})_{i,j=1}^n}.$$

Using this definition of the Schur functions, we see that if $p_k s_\lambda = \sum_\mu \chi^k_{\mu/\lambda} s_\mu$ for $k > 0$ then

$$B_{-k} \cdot G(s(\lambda)) \equiv \sum_\mu \chi^k_{\mu/\lambda} G(s(\mu)) \mod qL(\mathcal{F}).$$

So modulo $qL(\mathcal{F})$ the action of the Bosonic operators B_{-k} on normally ordered wedges is the same as multiplication by the power sums p_k on the Schur basis in the ring of symmetric functions. Thus $S_\lambda \cdot G(s^\circ = s(\emptyset)) = G(s(\lambda))$.

□

Let $\tilde{s} = (b_m, b_{m+1}, \ldots)$ be any normally ordered sequence satisfying $H(b_i \otimes b_{i+1}) = 1$ for all i and define $\tilde{s}(\lambda) := (z^{-\lambda_1} b_m, z^{-\lambda_2} b_{m+1}, \ldots)$. Assuming the existence of the global basis $\{\mathbb{G}_s\}$, the above proof generalises to give

$$S_\lambda \cdot \mathbb{G}_{\tilde{s}} = \mathbb{G}_{\tilde{s}(\lambda)}. \tag{5.10}$$

This is a direct generalisation of Theorem 6.9 of [40]. The condition for the partition $\lambda^{(0)}$ in Theorem 6.9 of [40] to be n-restricted corresponds to our condition for $\tilde{s}$ to satisfy $H(b_i \otimes b_{i+1}) = 1$. As a consequence of (5.10), one would have the equality $c^\lambda_{s/\tilde{s}}(q) = \langle \mathbb{G}_{\tilde{s}(\lambda)}, G(s) \rangle$.

5.5 Ribbon functions of higher level

Kashiwara, Miwa, Petersen and Yung also give constructions of higher level q-deformed Fock spaces for the quantized affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. This is generalised in the work of Takemura and Uglov [58] who construct higher level q-deformed Fock spaces for all the affine algebras $U_q(\widehat{\mathfrak{sl}}_n)$, using semi-infinite wedges. These Fock spaces are equipped with a representation of the Heisenberg algebra but the Heisenberg algebra is no longer the full commutator of the action of a quantum algebra, as follows.

Let $N \geq 1$ and $L \geq 1$ be two integers. Takemura and Uglov define an action of $H \otimes U'_q(\widehat{\mathfrak{sl}}_N) \otimes U'_q(\widehat{\mathfrak{sl}}_L)$ on a Fock space $\mathcal{F}$. The action of $U'_q(\widehat{\mathfrak{sl}}_N)$ is of level L and the action of $U'_q(\widehat{\mathfrak{sl}}_L)$ if of level N. This double action of quantized affine algebras is an example of *level-rank duality*. When $L = 1$, we obtain the Fock space representation of [28]. Hence the *higher level ribbon functions* $\{F_s\}$ defined using this action of H on $\mathcal{F}$ are natural generalisations of ribbon functions.

The parameters of this action of the Heisenberg algebra have not been computed fully, but it is conjectured that ([58, Conjecture 4.6])

$$a_i = \frac{1-q^{2Ni}}{1-q^{2i}}\frac{1-q^{2Li}}{1-q^{2i}}.$$

The double action of quantum affine algebras should be thought of as part of the action of a *quantum toroidal algebra*; see [15]. This quantum toroidal algebra is the "Schur-dual" to Cherednik's double affine Hecke algebra ([59]), so this may eventually give a representation theoretic explaination of the connection between Macdonald polynomials and ribbon functions in [16].

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