Math 419 Section 3 – Winter 2000
Final Review Problems

1. If A is an $n \times m$ matrix such that A^tA and AA^t are both invertible, must $m = n$? Explain.

2 Given a 3×3 matrix A such that $A^3 = I$, must all eigenvalues of A be real? Explain.

3. Do the matrices $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$ have the same image? Explain.

4. For a square matrix A is it always true that $\text{Im}(A^3 + A^2) \subset \text{Im}\,(A)$? Justify your answer.

5. In this problem, $A = \begin{pmatrix} 1 & 1 & 0 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 6 & 0 & 9 \\ 0 & 1 & 0 & 2 & 1 & 11 \end{pmatrix}$.

1. Find the row echelon form of A. Do not use technology.
2. Find a basis for the kernel of A.
3. Find a basis for the image of A.
4. Find the set of all solutions to: $A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

6. Let W be the image of $\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$. Find a basis for $W^\perp$ and find the matrices of the orthogonal projections onto W and $W^\perp$.

7. If A and B are matrices which are identical except for one column, find a formula for $\det(rA + (1-r)B)$ in terms of $\det(A)$, $\det(B)$, n and the scalar r.

8. Let A be a 3×3 symmetric matrix. Can the kernel of $A^2 + A + I$ be non-zero? Prove your answer correct.

9. Let $\vec{a}$ be a non-zero vector in $\mathbb{R}^4$, and let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation: $T\vec{x} = \vec{x} - 2(\vec{x} \cdot \vec{a})\vec{a}$. Find the eigenvalues and eigenvectors of T. Do not compute the matrix of T; reason geometrically.

10. Find the least squares solution to: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}\vec{x} = \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix}$.

11. In this problem, $A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \\ 2 & -2 & 4 & -4 \\ -2 & 2 & -4 & 4 \end{pmatrix}$ **(a)** Find a basis for the image of A. **(b)** Find a basis for the kernel of A. **(c)** Find all eigenvalues and eigenvectors of A.

12. Find the inverse of $A = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$.

13. For real matrices A: **(a)** Is $\ker(A) \subset \ker(A^tA)$? Explain why or give a counterexample. **(b)** Is $\ker(A) \supset \ker(A^tA)$? Explain why or give a counterexample.

14. Find the QR decomposition for $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$.

15. Which of the following matrices are similar? Explain! **(a)** $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$; **(b)** $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$; **(c)** $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

16. Compute the determinant of the matrix of No. 12, by the method of Gaussian elimination. Show your work.