Semiclassical Spectral Estimates for Toeplitz Operators

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Contents

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1 Introduction and Statement of Results

Let $P = \partial W$ be the boundary of a compact, smooth strictly pseudoconvex domain W, and let α be the contact form associated to a defining function r that is positive in the interior of the domain and zero on P. Specifically,

$$\alpha \;=\; j^*\mathrm{Im}\overline{\partial r} \tag{1}$$

where $j : P \to W$ is the inclusion map. Let $dp = \alpha \wedge (d\alpha)^n/2\pi$ denote the natural volume form, where the dimension of P is $2n+1$. The manifold

$$Z \;:=\; \{\, (p,\, r\alpha_p)\;;\; p \in P\; r > 0\,\} \tag{2}$$

is a symplectic submanifold of $T^*P \setminus \{0\}$ that serves as phase space for the algebra of Toeplitz operators on P, see [6] and [8]. Recall that a Toeplitz operator is an operator of the form

$$T \;=\; \Pi\, Q \;:\; \mathcal{H} \to \mathcal{H} \tag{3}$$

where $\mathcal{H} \subset L^2(P)$ is the Hardy space, $\Pi \;:\; L^2(P) \to \mathcal{H}$ the Szegö projector, [7], and Q a (classical) pseudodifferential operator on P. The symbol of (3), σ_T, is the restriction of the symbol of Q to Z, and T is called elliptic if this symbol is nowhere zero, [8]. Let us assume that this is the case, in fact that $\sigma_T > 0$, and furthermore that Q is of order one and self-adjoint. Then T is bounded below and its spectrum is a discrete set of eigenvalues,

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots . \tag{4}$$

In [8], §12, Boutet de Monvel and Guillemin show that $\sum_j e^{-it\lambda_j}$ is a tempered distribution and study its singularities, which are governed by the periodic trajectories of the Hamilton flow of σ_T.

Let V denote the contact vector field on P, uniquely determined by the conditions: $\alpha\rfloor V = 1$, $d\alpha\rfloor V = 0$. The case when V generates a free S^1 action of automorphisms of P is particularly interesting in semi-classical analysis, as we will discuss shortly ($S^1 = \mathbb{R}/2\pi\mathbb{Z}$). We will henceforth assume this to be the case, and accordingly denote V by ∂_θ. We are assuming that ∂_θ generates a flow of *automorphisms* of P, that is, that the flow preserves all structures and in particular commutes with the $\overline{\partial}_b$ complex. Therefore ∂_θ commutes with Π. Let

$$\mathcal{H} = \bigoplus_k \mathcal{H}_k \tag{5}$$

be the decomposition of $\mathcal{H}$ into eigenspaces of ∂_θ:

$$\partial_\theta|_{\mathcal{H}_k} = ik\,, \quad i = \sqrt{-1}\,, \tag{6}$$

and assume that the multiplier, Q, in (3) commutes with ∂_θ. We can then simultaneously diagonalize T and ∂_θ:

$$\left\{ \begin{array}{rcl} T\,\psi_i^k &=& kE_i^k\,\psi_i^k \\ \partial_\theta\,\psi_i^k &=& ik\,\psi_i^k \end{array} \right. \tag{7}$$

(the reason for denoting the joint eigenvalues by kE_i^k will become clear in a moment). In this paper we will study the asymptotic behavior of the $(E_i^k,\,\psi_i^k)$ as i and k tend to infinity in the regime: $|E_i^k - E| \le c/k$, where E and $c > 0$ are constants. More precisely we will study the trace of the Fourier coefficients of the operator $\varphi(T - E(-i\partial_\theta))$, where E is a constant and φ a smooth function with compactly supported Fourier transform. (This regime corresponds to the semi-classical analogue of the procedure of reduction of the classical Hamiltonian system (Z, σ_T) with respect to the S^1 action, [18].) We now discuss how this setting and problem arise in semi-classical analysis.

Semi-classical analysis in quantum mechanics usually consists of the study of spectral properties of (self-adjoint) pseudo-differential operators on a smooth manifold, in a small parameter limit. That is, one considers operators of the form $a(x, \hbar D_x)$, where the symbol $a(x, \xi)$ is a smooth function on the cotangent bundle T^*M of a Riemannian manifold M, the configuration space (see e.g. [24] for the case $M = \mathbb{R}^n$ and [21] for the Riemannian case). There are however many situations in physics where the classical underlying phase-space is not of the form T^*M, but is a Kähler manifold. Examples include spin models in nuclear physics (e.g. the liquid drop model), quantization of the two-dimensional torus (cat map [19], [11], the baker transformation [3]), perturbations of degenerate harmonic oscillator where the "secular equation" is solved by quantization of a reduced phase-space [15], and more generally semi-classical theories of quantum systems having symmetries. When the phase space is a Kähler manifold the associated Hilbert space is the space of square-integrable holomorphic sections of a quantizing line bundle, and the operators that naturally arise as quantization of observables are Toeplitz operators. We mention the pioneering work of F.A. Berezin [2] in recognizing the use of Toeplitz operators when the phase space is a Kähler manifold. We also mention that in [4] such Toeplitz operators were shown to satisfy deformation quantization conditions, meaning that they exhibit the semiclassical behavior one normally requires of a quantization scheme.

Thus let X be a compact Kähler manifold of real dimension $2n$, and $L \to X$ a holomorphic Hermitian line bundle such that the curvature of its natural connection is the Kähler form of X. (This implies that the Kälher form of X is integral.) The quantization of X with Planck's constant equal to $1/k$ is the space $H^0(X, L^{\otimes k})$, and the Toeplitz quantization of a classical Hamiltonian function, $H \in C^\infty(X)$, is the operator

$$S_k \;:=\; \pi_k \, M_H \, \pi_k \,,$$

where $\pi_k \,:\, L^2(X, L^{\otimes k}) \to H^0(X, L^{\otimes k})$ is the orthogonal projector and M_H is the operator of multiplication by H. We claim that this setting gives rise to the one at the beginning of this Introduction: Let $P \subset L^*$ be the unit circle bundle in the dual of L. By the condition that the curvature of L be the Kähler form, P is strictly pseudoconvex and the contact form α is the connection form on P. As is well-known, in the decomposition (5) of the Hardy space of P one can naturally make the identification

$$\mathcal{H}_k \;=\; H^0(X, L^{\otimes k})\,. \tag{8}$$

Notice that, by the Kodaira vanishing theorem, for large k the dimension of $\mathcal{H}_k$,

$$d_k \;:=\; \dim \mathcal{H}_k \tag{9}$$

is given by the Riemann-Roch theorem, and therefore for large k it is a polynomial in k of degree n with leading coefficient $\mathrm{Vol}(X)/(2\pi)^n$. Under the identification (8) the operator S_k gets identified with the operator $S_k = \Pi_k M_H \, \Pi_k$ where Π_k is the orthogonal projection

$L^2(P) \to \mathcal{H}_k$ and (by a slight abuse of notation) M_H the operator of multiplication by the pull-back of H to P (and therefore $[M_H,\, \partial_\theta] = 0$). It follows that if we consider the joint eigenvalues and eigenvectors, (7), of the Toeplitz operator

$$T \;=\; \Pi\, D_\theta\, M_H \Pi\,, \quad D_\theta \;:=\; \frac{1}{i}\,\partial_\theta \tag{10}$$

then, for each k, $(E_i^k,\, \psi_i^k)$, $i = 1, \dots d_k$, are the eigenvalues and eigenvectors of S_k:

$$\forall k = 1, 2, \dots \qquad S_k\, \psi_i^k \;=\; E_i^k\, \psi_i^k\,, \qquad i = 1, \dots d_k\,. \tag{11}$$

We now state our main results. We shall keep the notation of the previous paragraph: X is a quantized Kähler manifold, $H \in C^\infty(X)$ defining the operator S_k on $\mathcal{H}_k$ with eigenvalues and eigenfunctions $(E_i^k,\, \psi_i^k)$. Let ϕ be the Hamiltonian flow of H on X, and $\pi : P \to X$ the obvious projection. For both of the theorems stated below, φ denotes a smooth test function with compactly supported Fourier transform.

Our first result is concerned with the integral kernel of $\varphi(k(T_k - E))$ where $T_k \;=\; T|_{\mathcal{H}_k}$.

Theorem 1.1 *Let $p_1, p_2 \in P$ and let $x_j = \pi(p_j)$, $j = 1, 2$. In case $p_1 = p_2 = p$ we further assume that $x_1\, (= x_2)$ is not a fixed point of ϕ. Then:*

1. *If $H(x_1) = E = H(x_2)$, and x_1, x_2 lie on the same orbit of ϕ, then the sums*

$$\sum_{i=0}^{\infty} \varphi(k(E_i^k - E))\; \psi_i^k(p_1)\; \overline{\psi_i^k(p_2)} \tag{12}$$

admit an asymptotic expansion as $k \to \infty$ of the form

$$\sum_{\tau} \sum_{l=0}^{\infty} C_{\tau,l}\hat{\varphi}(\tau) e^{ik\theta_\tau} k^{n-1/2-l},$$

where the first summation is over the real numbers τ such that $\phi_\tau(x_2) = x_1$. The coefficient $C_{\tau,0}$ is computed in Theorem 2.7 and the angles θ_τ are defined in (23).

2. *If at least one of x_1, x_2 does not belong in $H^{-1}(E)$ or these points are not connected by any trajectory of ϕ, then (12) decreases rapidly in k.*

Remark: The sum (12) measures the correlation between two points of eigenfunctions in a band of of energies whose width decreases as $1/k$. Our result demonstrates a certain coherence among eigenfunctions at points which are related to each other classically, and shows that there is no correlation otherwise. For $p_1 = p_2$, the sum is the average of the probability distribution function over a narrow band of energies.

Our second result concerns the trace of $\varphi(k(T_k - E))$ and is the semi-classical trace formula (Gutzwiller's formula) adapted to the present setting (for a more detailed and more general statement see Theorem 4.2):

Theorem 1.2 *Under certain assumptions on cleanness of the fixed point sets of ϕ on $H^{-1}(E)$ (see §4), the following weighted trace of S_k admits an asymptotic expansion:*

$$\sum_{i=0}^{\infty} \varphi(k(E_i^k - E)) \;\sim\; \sum_{j \in J} \sum_{l=0}^{\infty} C_{j,l}(\varphi)\, e^{ik\theta_j}\, k^{(d_j-1)/2-l}, \tag{13}$$

where J indexes the connected components of the set of pairs $(x,\tau) \in H^{-1}(E) \times \mathbb{R}$ with $\phi_\tau(x) = x$, d_j denotes the dimension of the j-th component, and the angles θ_j are the holonomy angles for the closed trajectories. The coefficients $C_{j,0}(\varphi)$ are described in §4.

The proof of Theorem 1.1 appears in §2 and that of Theorem 1.2 in §4 where we also discuss Weyl-type estimates. §3 is devoted to the important case when X is an integral coadjoint orbit of a compact Lie group.

A Remark on Notation. The calculations we are about to embark on are plagued by a profusion of powers of 2π which depend only on the dimensions of the manifolds involved. Rather than keeping track of such powers at each step, we have opted to ignore them until the end. Thus in the *proofs* presented in sections 2 and 4 the displayed equations are to be taken modulo a (multiplicative) power of 2π. The correct constants in the Theorems are calculated by considering an example: the harmonic oscillator for Theorem 2.7 and the well-known trace formula for Theorem 4.2.

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2 Proof of Theorem 1.1

2.1 Existence of the expansion

We begin with some preliminaries. Let $H \in C^\infty(X,\mathbb{R})$ be the Hamiltonian on X, which we will consider as a function on P which is S^1 invariant, let M_H be the associated multiplication operator on $L^2(P)$ and let $S = \Pi M_H \Pi$. We are interested in the Toeplitz operator $D_\theta S = \Pi D_\theta M_H \Pi$. By Lemma 12.2 of [8] we can choose a self-adjoint first order pseudodifferential operator $Q : L^2(P) \to L^2(P)$ such that

$$\Pi Q \Pi = D_\theta S \qquad \text{and} \qquad [Q,\Pi] = 0. \tag{14}$$

We note that in many cases of physical interest, the line bundle L is trivialized off a divisor and therefore the space $\mathcal{H}_k$ can be identified with a space of analytic functions. The Hamiltonian and observables of interest are given as differential operators preserving analyticity. *This means that often the operator Q above is given at the outset.* We are given in particular its principal and subprincipal symbols. The former of course is the corresponding classical observable and the latter is related to a choice of operator ordering.

For $t \in \mathbb{R}$ define the operator

$$B = e^{-it\Pi D_\theta M_H \Pi}\Pi = e^{-itQ}\Pi, \tag{15}$$

fundamental solution of the equation on $\mathcal{H}$, $D_t\psi = -Q\psi$. In terms of the eigenvalues and eigenfunctions of S, the Schwartz kernel of this operator, considered as an operator from $C^\infty(P)$ to $C^\infty(\mathbb{R}\times P)$, is

$$\mathcal{B}(t,p,q) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e^{-iktE_j^k}\ \psi_j^k(p)\ \overline{\psi_j^k(q)}. \tag{16}$$

Define also the operator $F : C_0^\infty(\mathbb{R}\times P\times P) \to C_0^\infty(S^1)$ by

$$F : f(t,p,q) \mapsto \int e^{i\kappa(s+tE-\theta)}\hat{\varphi}(t) f(t, R_\theta(p_1), p_2)\, d\kappa\, d\theta\, dt, \tag{17}$$

where φ is the test function and $p_1, p_2 \in P$ are two points of P (the ones appearing in the statement of the theorem), and R denotes the natural action of S^1 on P. Using the fact that $\hat{\varphi}$ has compact support, we can extend the domain of F to include distributions such as $\mathcal{B}$. Finally, define $\Upsilon = F(\mathcal{B})$. Explicitly,

$$\Upsilon(s) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \varphi(k(E_i^k - E))\, \psi_i^k(p_1)\, \overline{\psi_i^k(p_2)}\, e^{iks}. \tag{18}$$

Observe that the k-th Fourier coefficient of Υ, namely

$$\sum_{i=0}^{\infty} \varphi(k(E_i^k - E))\, \psi_i^k(p_1)\, \overline{\psi_j^k(p_2)} \tag{19}$$

is precisely the k-th term of the sequence appearing in the left-hand side of Theorem 1.1.

The strategy of the proof of the existence of the asymptotic expansion is the following. We will prove that B is an FIO of Hermite type, see [16], that F is a standard FIO (this is practically obvious), and will verify the hypotheses of the composition theorem of [8]. It will follow that Υ is a Lagrangian distribution on the circle, which immediately implies the existence of the asymptotic expansion of its Fourier coefficients.

We now proceed with the details. Recall that $Z \subset T^*P$ by is the symplectic submanifold $Z = \{(p, r\alpha_p); p \in P, r > 0\}$, where α is the contact form. T^*P possesses an $\mathbb{R}^+$ action, whose generator is denoted ∂_r. T^*P also inherits an S^1 action from the S^1 action on P, which we again denote by R_θ, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Let ∂_Θ be the infinitesimal generator of this action. There is a Hamiltonian function $J : T^*P \to \mathbb{R}$ generating this action, which restricted to Z is given by $J(p, r\alpha_p) = r$.

Let ρ denote the projection $Z \to P$. To H we associate the Hamiltonian $\tilde{H} : Z \to \mathbb{R}$ given by

$$\tilde{H}(z) = J(z) H(\rho(z))\,. \tag{20}$$

Let $\{\phi_t\}$ denote the flow on X associated to H and $\{\tilde{\phi}\}$ the flow on Z associated to $\tilde{H}$, with Ξ and $\tilde{\Xi}$ the respective infinitesimal generators. Note that $\tilde{H}$ is invariant under the S^1 action and homogeneous of degree 1 under the $\mathbb{R}^+$ action. In particular, $\tilde{\phi}_t$ preserves J. We can also define a "horizontal" flow on Z by lifting the flow ϕ_t to P horizontally, and then extending this to a flow ϕ_t^h on Z by requiring that J be preserved. Let Ξ_h be the infinitesimal generator of this flow. A simple calculation shows that

$$\phi_t^h(z) = R_{-tH(\rho(z))}(\tilde{\phi}_t(z)) \tag{21}$$

(where again R_θ denote the natural action of S_1 on P) and, correspondingly,

$$\Xi_h = \tilde{\Xi} - H\partial_\Theta. \tag{22}$$

By the homogeneity of $\tilde{H}$, $\omega(\tilde{\Xi}, \partial_r) = d\tilde{H}(\partial_r) = H$, but $\omega(\Xi_h, \partial_r) = 0$. At this point we can explain the angles θ_τ appearing in Theorem 1.1. If τ is such that $\phi_\tau(\pi(p_2)) = \pi(p_1)$, then there exists θ_τ such that

$$R_{\theta_\tau}(\phi^h(p_2)) \;=\; p_1\,. \tag{23}$$

Our first result involves the moment Lagrangian:

$$\Sigma = \{(t, \tilde{H}(z); \tilde{\phi}_t(z); z);\; z \in Z, t \in \mathbb{R}\} \subset T^*\mathbb{R} \times Z^- \times Z. \tag{24}$$

Note that Σ is an isotropic submanifold when considered as a subspace of $T^*(\mathbb{R}\times P\times P)$.

Our first results will involve the spaces of Hermite distributions of [8]. Recall that in *op.cit.*, Boutet the Monvel and Guillemin define spaces of distributions $I^l(M,C)$ on a $\Leftarrow$ manifold M, for any conic isotropic submanifold $C\subset T^*M\setminus\{0\}$. In case C is actually a Lagrangian submanifold, these spaces coincide with Hörmander's spaces of Lagrangian distributions, [20]. There is however a discrepancy in the definition of the order: here we use the convention of [8].

Lemma 2.1 *The Schwartz kernel, T, of the fundamental solution of the equation $D_t\psi = -Q\psi$, (16), is in the space $I^{1/2}$ of [8]:*

$$\mathcal{B}\in I^{1/2}(\mathbb{R}\times P\times P,\ \Sigma). \tag{25}$$

PROOF Let $\mathcal{A}$ denote the Schwartz kernel of the operator e^{-itQ} and $\mathcal{K}$ denote the kernel of Π. Thus $\mathcal{B}=\mathcal{A}\circ\mathcal{K}$. By [13], Theorem 1.1,

$$\mathcal{A}\in I^{n+1/2}(\mathbb{R}\times P\times P,\ \mathcal{C}), \tag{26}$$

where $\mathcal{C}$ is the moment Lagrangian for the flow of the symbol of Q on T^*P. As mentioned here we use the convention of [8] for the order of a Lagrangian distribution, which in the present case differs from that of Hörmander by the addition of $(\dim\mathbb{R}\times P\times P)/4$. We also have, by Theorem 11.1 of [8],

$$\mathcal{K}\in I^{1/2}(P\times P, Z\stackrel{\wedge}{\times} Z), \tag{27}$$

where $Z\stackrel{\wedge}{\times} Z := \{(z,z)\,;\ z\in Z\}$. The fiber product diagram

$$\begin{array}{ccc} \mathcal{F} & \rightarrow & \mathcal{C}\times(Z\stackrel{\wedge}{\times} Z) \\ \downarrow & & \downarrow \\ T^*P\stackrel{\wedge}{\times} T^*P & \hookrightarrow & T^*P\times T^*P \end{array} \tag{28}$$

is clean with zero excess (see [8], page 44, for definitions). Also, because of (14) the restriction of the symbol of Q to Z is given by the function $\tilde{H}(p,r\alpha_p)=rH(p)$. Thus, $\mathcal{C}\circ(Z\stackrel{\wedge}{\times} Z)=\Sigma$, and by Theorem 9.5 of [8],

$$\mathcal{A}\circ\mathcal{K}\in I^{(n+1/2)+1/2-(\dim P)/2}(\mathbb{R}\times P\times P,\Sigma), \tag{29}$$

yielding the result above.

$\square$

For $j=1,2$ define $L_j=T^*_{p_j}P$. Note that each L_j is a Lagrangian submanifold of T^*P, and that $L_j\cap Z$ is one-dimensional with tangent space spanned by ∂_r. Define the Lagrangian relation

$$\begin{aligned} \Gamma &= \{(\theta-tE, J(m_1);t,EJ(m_1);R_\theta(m_1);m_2); t\in\mathbb{R},\theta\in\mathbb{R}/2\pi\mathbb{Z}, m_i\in L_i\} \\ &\subset T^*S^1\times(T^*\mathbb{R}\times T^*P\times T^*P). \end{aligned} \tag{30}$$

Lemma 2.2 *The Schwartz kernel of F, $\mathcal{K}_F$, is in the space*

$$\mathcal{K}_F\in I^{2n+1}(S^1\times\mathbb{R}\times P\times P,\Gamma). \tag{31}$$

PROOF We can write $\mathcal{K}_F$ locally as the oscillatory integral

$$\mathcal{K}_F(s,t,p,q) = \hat{\varphi}(t)\int e^{i\psi(s,t,p,q,\kappa,\theta,\xi,\xi')}d\kappa\, d\theta\, d\xi\, d\xi', \tag{32}$$

where

$$\psi(s,t,p,q,\kappa,\theta,\xi,\xi') = \kappa(s+tE-\theta) + \xi\cdot(p-R_\theta(p_1)) + \xi'\cdot(q-p_2). \tag{33}$$

To find the associated Lagrangian relation we first define the critical set

$$C_\psi = \{(s,t,p,q,\kappa,\theta,\xi,\xi');\, d_\kappa\psi = d_\theta\psi = d_\xi\psi = d_{\xi'}\psi = 0\}. \tag{34}$$

The Lagrangian relation is the image of the map $C_\psi \to T^*(S^1\times\mathbb{R}\times P\times P)$ given by

$$(s,t,p,q,\kappa,\theta,\xi,\xi') \mapsto (s,t,p,q;\frac{\partial\psi}{\partial s},\frac{\partial\psi}{\partial t},\frac{\partial\psi}{\partial\xi},\frac{\partial\psi}{\partial\xi'}). \tag{35}$$

In our case,

$$C_\psi = \{(\theta-tE,t,p_1\cdot e^{i\theta},p_2;\kappa,\theta,\xi,\xi')\}, \tag{36}$$

and the relation is easily seen to be the Γ defined above.

In computing the order of $\mathcal{K}_F$ as a distribution, there is a slight technicality because ψ is not homogeneous in the phase variables. To remedy this problem, we introduce a new variable $\alpha = \kappa\theta$. Then ψ is homogeneous of degree one as a function of (κ,α,ξ,ξ'), and

$$\mathcal{K}_F(s,t,p,q) = \hat{\varphi}(t)\int e^{i\psi(s,t,p,q;\kappa,\alpha,\xi,\xi')}\kappa^{-1}d\kappa\, d\alpha\, d\xi\, d\xi'. \tag{37}$$

The relationship between the order of the distribution, m, and the degree of homogeneity of the amplitude, s, is $m - N/2 = s$, where N is the number of phase variables. So in our case, $m = -1 + (4n+4)/2 = 2n+1$.

□

Proposition 2.3 *The composition $\Gamma\circ\Sigma$ is non-empty if and only if p_1 and p_2 both lie in the energy surface $H^{-1}(E)$ and $\pi(p_1) = \phi_\tau(\pi(p_2))$ for some τ. If these two conditions are satisfied, then*

$$\Gamma\circ\Sigma = \{(\theta,r);\, \phi^h_\tau(p_2;r\alpha_{p_2}) = R_\theta(p_1;r\alpha_{p_1})\} \subset T^*S^1. \tag{38}$$

PROOF The proof is essentially immediate. The only point to make is that the the singularities occur at points $\theta - \tau E$ for the flow $\tilde{\phi}_\tau$, which becomes simply the rotation angle θ under the flow ϕ^h_τ, in view of (21).

□

Corollary 2.4 *If p_1 and p_2 are not connected by a trajectory of Ξ, then the sum (19) decreases rapidly in k.*

PROOF This follows because $WF(\Upsilon)\subset\Gamma\circ\Sigma$, and if this is empty the Fourier coefficients of Υ are rapidly decreasing.

□

Proposition 2.5 *The fiber product diagram*

$$\begin{array}{ccc} \mathcal{F} & \rightarrow & \Gamma \\ \downarrow & & \downarrow \rho \\ \Sigma & \hookrightarrow & T^*\mathbb{R} \times T^*P \times T^*P \end{array} \tag{39}$$

is clean provided $\pi(p_1)$ is not a fixed point of the flow of H on X. The excess of the diagram is $2n$. (Again we refer to [8], page 44, for definitions)

Remark If $p_1 \neq p_2$, and one of the two is a fixed point, then there does not exist a τ such that $\pi(p_1) = \phi_\tau(\pi(p_2))$, so $\Gamma \circ \Sigma = \emptyset$ in this case. Thus the only situation for which the cleanness becomes a restriction occurs when $p_1 = p_2$ is a fixed point.

PROOF Recall that definition of the fiber product is that

$$\mathcal{F} = \{(\gamma, \sigma) \in \Gamma \times \Sigma;\ \rho(\gamma) = \sigma\}. \tag{40}$$

The two conditions defining a clean fiber product diagram are that $\mathcal{F}$ be a submanifold of $\Gamma \times \Sigma$ and that the linearized diagram

$$\begin{array}{ccc} T_{\gamma,\sigma}\mathcal{F} & \rightarrow & T_\gamma\Gamma \\ \downarrow & & \downarrow d\rho_\gamma \\ T_\sigma\Sigma & \hookrightarrow & T_\sigma(T^*\mathbb{R} \times T^*P \times T^*P) \end{array} \tag{41}$$

where $\gamma \in \Gamma, \sigma \in \Sigma$, and $\rho(\gamma) = \sigma$, must again be a fiber product diagram. In the diagram (39), we have

$$\begin{aligned} \mathcal{F} \;=\; & \{((\theta - \tau E, J(z); \tau, EJ(z); \tilde{\phi}_\tau(Z); z), (\tau, H(z); \phi_\tau(z); z)); \\ & H(z) = EJ(z), z \in L_2 \cap Z, \phi_\tau(z) \in R_\theta(L_1)\}, \end{aligned} \tag{42}$$

which is clearly a submanifold of $\Gamma \times \Sigma$. Let $(\gamma, \sigma) \in \mathcal{F}$, such that the image of γ under the natural projection $\Gamma \to T^*S^1$ is $(\theta - \tau E; 1)$. For convenience, we adopt the notation

$$\check{\Gamma} = T_\gamma\Gamma, \qquad \check{\Sigma} = T_\sigma\Sigma, \tag{43}$$

and similarly denote the tangent spaces of Z and L_j. Since $\mathcal{F}$ is one dimensional, to prove the second cleanness condition it suffices to show that $(T_{(\theta-\tau E;1)}(T^*S^1) \oplus \check{\Sigma}) \cap \check{\Gamma}$ is one dimensional. Now,

$$\check{\Sigma} = \{(s, dH(v); s\tilde{\Xi} + d\tilde{\phi}_\tau(v); v);\ v \in \check{Z}, s \in \mathbb{R}\}, \tag{44}$$

and

$$\check{\Gamma} = \{(\kappa - sE, dJ(\mu_1); s', EdJ(\mu_1); \kappa\partial_\Theta + dR_\theta(\mu_1); \mu_2); \qquad \kappa, s' \in \mathbb{R}, \mu_j \in \check{L}_j\}. \tag{45}$$

Thus $(T_{(\theta-TE;1)}(T^*S^1) \oplus \check{\Sigma}) \cap \check{\Gamma}$ is defined by the conditions:

$$s = s', \qquad dH(v) = E\, dJ(\mu_1), \qquad s\Xi + d\phi_\tau(v) = dR_\theta(\mu_1) + \kappa\partial_\Theta, \qquad v = \mu_2. \tag{46}$$

The last condition implies $v = \mu_2 = a\partial_r$ for some $a \in \mathbb{R}$. Note that $dR_\theta(\partial_r) = d\tilde{\phi}_\tau(\partial_r) = \partial_r$. Thus $\mu_1 = a\partial_r$ also. The second condition is thus satisfied. The third condition becomes $s\Xi = \kappa\partial_\Theta$. Since $\Xi \neq 0$ at p_1 by assumption, this implies $\kappa = s = 0$ (and thus $s' = 0$ also by the first condition). Therefore the intersection is one-dimensional.

$\square$

Lemma 2.6 *The distribution, Υ, is a Lagrangian distribution on the circle. More precisely*

$$\Upsilon(s) \in I^n(S^1, \Gamma \circ \Sigma). \tag{47}$$

Proof The result follows directly from Proposition 2.5 and Theorem 7.5 of [8].

□

Theorem 1.1 is an immediate corollary of this theorem.

2.2 Calculation of the leading coefficient

In this section will state and prove a theorem describing the coefficient $C_{\tau,0}$. We must first deal with a few preliminaries. To a symplectic vector space V, we associate a Hilbert space $H(V)$ and subspace $H_\infty(V) \subset H(V)$ as follows. Define $H(V)$ to be the representation space of the Stone-von Neumann representation of $V \oplus \mathbb{R}$ as a Heisenberg algebra (for general background on this see [14]). Let $H_\infty(V)$ be the set of smooth vectors for this representation. $H(V)$ can be identified with $L^2(\mathbb{R}^{\dim V/2})$, and $H_\infty(V)$ with the space of Schwartz functions $\mathcal{S}(\mathbb{R}^{\dim V/2})$. We denote the dual of $H_\infty(V)$, identified with the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{\dim V/2})$, by $H_\infty(V)'$.

Let $T_p^h P$ denote the horizontal subspace of T_pP (by definition the null-space of α). $T_p^h P$ is naturally identified with $T_{\pi(p)}X$, so the Kähler structure on X gives us a positive definite Lagrangian subspace Λ_p of $T_p^h P \otimes \mathbb{C}$, namely the type $(1,0)$ subspace. We can think of this subspace as an abelian subalgebra of the complexified Heisenberg algebra, and so it acts on $H_\infty(T_p^h P)'$, by extension of the Stone-von Neumann representation. The kernel of the action of a positive definite Lagrangian subspace is, by Proposition 4.2 of [8], one-dimensional, and in fact consists only of smooth vectors. We will denote it by $\mathcal{W}_p$:

$$\mathcal{W}_p := \ker \Lambda_p \text{ in the Stone-von Neumann representation } \subset H_\infty(T_p^h P)\,.$$

This subspace is uniquely determined by the Kähler structure of X, and it essentially defines the symbol of the Szegö projector Π as a Hermite FIO. More precisely, the symbol of the Szegö projector at $(\alpha_p\,, \alpha_p)$ can be identified with the orthogonal projection,

$$H(T_p^h P) \to \mathcal{W}_p\,,$$

see Theorem 11.2 in [8]; for the expression of the symbol of the Szegö projector in the language of Fourier integral operators with complex phase see [9] . We note that *there is not, in general, a natural choice of normalized generator of the subspaces $\mathcal{W}_p$, $p \in P$.* (We will see however in §3 examples where it *is* possible to make a canonical choice of generators.) The possible choices for each point p form a circle bundle over P, corresponding to the possible choices of phases of normalized generators of $\mathcal{W}_p$.

Let $H \in C^\infty(P)$ be S^1-invariant, Q be the pseudodifferential operator of (14), and more generally resume the notation of §2.1. Let p_1, p_2 be two points on P such that $\phi_\tau^h(p_2) = p_1$, where ϕ_t^h is the horizontal lift of the Hamilton flow of H considered as a function on X. To state a formula for the coefficient $C_{\tau,0}$ of Theorem 1.1 we need to fix a normalized section of the bundle $\mathcal{W} := \cup_{p\in P}\mathcal{W}_p$ over the trajectory of ϕ_t^h joining p_2 to p_1. Denote the value of this section at p by e_p. The differential of the flow on X gives a symplectic map $d\phi_\tau : T_{p_2}^h P \to T_{p_1}^h P$. By continuity we lift this to an operator in the metaplectic representation,

$M_\tau : H_\infty(T^h_{p_2}P) \to H_\infty(T^h_{p_1}P)$. Let Ξ be the infinitesimal generator of the flow on X. Then Ξ acts on $H(T^h_pP)$ (and hence $H_\infty(T^h_pP)$) via the Heisenberg representation. The projection from $H_\infty(T^h_pP)$ to (generalized) vectors invariant under Ξ is defined by

$$P_\Xi(v) = \int_{-\infty}^{\infty} e^{it\Xi} v \, dt \,.$$

With these ingredients we can define a function, $c(t)$, on points of the trajectory, as follows. Let q be the symbol of Q, which generates a flow on T^*P. Since $[\Pi, Q] = 0$ this flow preserves Z and the symplectic normal of Z, which is canonically isomorphic to T^hP. Under this identification, the flow of q acts on $T^hP \otimes \mathbb{C}$ in such a way as to map Λ_p to $\Lambda_{\phi_t(p)}$. Thus the corresponding metaplectic map takes e_p to a multiple of $e_{\phi_t(p)}$. We can thus differentiate the section e_p by Ξ_q, the infinitesimal generator of this flow. We then define $c(t)$ by:

(48) $$\Xi_q e_{\phi_t(p)} = ic(t) e_{\phi_t(p)} \,.$$

Theorem 2.7 *With notation as above, the first term in the expansion is*

(49) $$C_{\tau,0} \;=\; \frac{1}{2\pi^{n+1}} \langle M_\tau^{-1} e_{p_1}, P_\Xi(e_{p_2})\rangle \; \exp\Big\{-i\int_0^\tau (\sigma_{sub}[Q] + c)\, dt\Big\},$$

where $\sigma_{sub}[Q](t) = \sigma_{sub}[Q](\tilde{\phi_t}(p_2, \alpha_{p_2}))$.

Remarks.

1. Once can check directly that the expression (49) is independent of the choice of section e. Indeed the phase dependence of the inner product in (49) on the choice of section e is cancelled by the exponential of the integral of $c(t)$. Thus the final result only depends on Q and the path.

2. The appearance of the subprincipal symbol of Q is the manifestation of a quantum mechanical ordering problem. The prescription for obtaining Q from H reflects the anti-Wick ordering induced in quantization by Toeplitz operators. As we have remarked above, however, in the physical situations we typically start from a knowledge of Q rather than H. (This is the point of view adopted in §3.)

3. In certain cases there exists a global normalized section e of the complex line bundle, $\mathcal{W} = \cup_{p\in P}\mathcal{W}_p$, and then the function

   (50) $$\sigma_{sub}[Q] + c$$

   (where c is defined as above) is defined in [8] ((11.6)) to be the Toeplitz operator subprincipal symbol of T, where $T = \Pi D_\theta M_Q \Pi = \Pi Q \Pi$. We will see in Section 3 a situation where a global choice of e is possible, see Lemma 3.5. When the section e exists globally, the function $\sigma_{sub}[Q] + c$ only depends on the choice of section e and on the Toeplitz operator T.

We now turn to the proof of Theorem 2.7, which amounts to the computation of the symbol of Υ at a point of $\Gamma \circ \Sigma$. Linearize the fiber product diagram at a particular point of Σ and Γ as in the proof of Proposition 2.5. The symbol of Υ at the point of localization is a half-form on $\check{\Gamma}\circ\check{\Sigma}$. Since this space is naturally identified with $\mathbb{R}$, it possesses a canonical half-form. The leading coefficient (for a particular τ) in the expansion is just the symbol of Υ divided by this half-form.

Let us consider again the Schwartz kernel $\mathcal{B}$ of the fundamental solution of the equation $D_t\psi = -Q\psi$ as in §2.1. The symbol of $\mathcal{B}$ is a symplectic spinor on $\check{\Sigma}$. Recall that the space of symplectic spinors associated to an isotropic subspace $\check{\Sigma}$ is given by [16]

$$\mathrm{Spin}(\check{\Sigma}) = \bigwedge{}^{1/2}\check{\Sigma}\otimes H_\infty(\check{\Sigma}^\perp/\check{\Sigma}). \tag{51}$$

The symbol of F is simply a half-form on $\check{\Gamma}$, as $\check{\Gamma}$ is Lagrangian.

The calculus of Hermite symbols used to compute the symbol of Υ is described in the first section of the Appendix. Let $W = T_\sigma(T^*(\mathbb{R}\times P\times P))$, and define U_0, U_1, U as in Definition A.1. U is isotropic by construction. The dimension of U_1 is the excess of the diagram (39), which is $2n$. The dimension of U_0 is the dimension of the fiber of the projection of $\mathcal{F}$ through Γ to T^*S^1, which is zero. Thus $U_0 = \{0\}$, implying $\dim U = 2n$ also. Now, $\dim\Sigma = 2n+3$ and $\dim W = 8n+6$, so $\dim\check{\Sigma}^\perp/\check{\Sigma} = 4n$. Therefore, U is Lagrangian in $\check{\Sigma}^\perp/\check{\Sigma}$. The Kostant map yields the pairing

$$\bigwedge{}^{1/2}U\otimes H_\infty(\check{\Sigma}^\perp/\check{\Sigma})\to\mathbb{C}. \tag{52}$$

The symbol map in the present case is therefore slightly simpler than the more general version given in the Appendix. The exact sequence (126) gives a half-form map:

$$\bigwedge{}^{1/2}\check{\Gamma}\otimes\bigwedge{}^{1/2}\check{\Sigma}\to\bigwedge{}^{1/2}(\check{\Gamma}\circ\check{\Sigma})\otimes\bigwedge{}^{1/2}U, \tag{53}$$

which when combined with the Kostant pairing yields the full symbol map

$$\bigwedge{}^{1/2}\check{\Gamma}\otimes\bigwedge{}^{1/2}\check{\Sigma}\otimes H_\infty(\check{\Sigma}^\perp/\check{\Sigma})\to\bigwedge{}^{1/2}(\check{\Gamma}\circ\check{\Sigma}). \tag{54}$$

To describe the symbol of $\mathcal{B}$, it is useful to make the following identification:

Lemma 2.8 *There is a natural symplectomorphism*

$$\check{\Sigma}^\perp/\check{\Sigma}\cong T^h_{p_1}P\times T^h_{p_2}P, \tag{55}$$

where T^hP denotes the horizontal tangent space of P (the null-space of α).

PROOF The spaces $\check{\Sigma}$, $\{(0,0;w;0);\ w\in d\phi_\tau(\check{Z})^\perp\}$, and $\{(0,0;0;w);\ w\in\check{Z}^\perp\}$ are linearly independent, and their total dimension adds up to $6n+3 = \dim W - \dim\check{\Sigma}$. Therefore,

$$\check{\Sigma}^\perp\cong\check{\Sigma}\oplus d\phi_\tau(\check{Z})^\perp\oplus\check{Z}^\perp. \tag{56}$$

It is straightforward to see (for instance by using Darboux coordinates), that the projection of $\check{Z}^\perp$ onto $T_{p_2}P$ is a symplectomorphism from $\check{Z}^\perp\to T^h_{p_1}P$. The corresponding statement of course holds true for $d\phi_\tau(\check{Z})^\perp$.

□

Let e_p be a section chosen as described at the beginning of this subsection. The symbol of Π at p_2 is

$$e_{p_2}\otimes\overline{e_{p_2}}\otimes\sqrt{d\check{Z}}\in H_\infty(T^h_{p_2}P)\otimes H_\infty(T^h_{p_2}P)\otimes\bigwedge{}^{1/2}(\check{Z}\,\hat{\times}\,\check{Z}), \tag{57}$$

where, abusing notation slightly, $d\check{Z}$ denotes the volume form that $\check{Z}\,\hat{\times}\,\check{Z}$ inherits from the symplectic structure on $\check{Z}$. Note that $\check{\Sigma}$ possesses a natural volume form which is simply the combination of the Liouville form on $\check{Z}$ with $d\tau$.

Proposition 2.9 *Under the identification $H_\infty(\check{\Sigma}^\perp/\check{\Sigma}) \cong H_\infty(T^h_{p_1}P) \otimes H_\infty(T^h_{p_2}P)$, the symbol of $\mathcal{B}$ at the point of linearization is*

$$(58)\quad \exp\Big\{-i\int_0^\tau(\sigma_{sub}[Q]+c)dt\Big\}\sqrt{d\check{\Sigma}}\otimes e_{p_1}\otimes\overline{e_{p_2}} \in \wedge^{1/2}\check{\Sigma}\otimes H_\infty(T^h_{p_1}P)\otimes H_\infty(T^h_{p_2}P).$$

PROOF The operator B satisfies the equation

$$(59)\qquad \Big[-i\frac{\partial}{\partial t}+Q\Big]B=0,$$

with initial data Π. The symbol of B is completely determined by the associated transport equation, see [8], with initial data the symbol of Π. This yields the result.

□

To define the symbol of F, note first that $L=T^*_{p_2}P$ possesses a half-form coming from the metric. Thus Γ possess a canonically defined half-form, which we label $\sqrt{d\Gamma}$.

Proposition 2.10 *The symbol of F at the point of linearization is*

$$(60)\qquad \hat{\varphi}(\tau)\sqrt{d\Gamma}\in\wedge^{1/2}\check{\Gamma}.$$

Lemma 2.11 *Under the identification given in Lemma 2.8,*

$$(61)\qquad U=\{(s\Xi_h+d\tilde{\phi}_\tau(v),v);\ v\in T^h_{p_1}\tilde{H}^{-1}(E), s\in\mathbb{R}\}.$$

PROOF We compute directly that

$$\begin{aligned}(62)\quad U_1 \;=\;& \{(s,0;s\tilde{\Xi}+d\phi^h_\tau(v)+w_1;v+w_2);v\in\check{Z},w_2\in\check{Z}^\perp,w_1\in d\tilde{\phi}_\tau(\check{Z})^\perp,\\ & d\tilde{H}(v)=dJ(v)=0,v+w_2\in L_2,s\Xi_h+d\tilde{\phi}_\tau(v)+w_1\in dR_\theta(L_1)\}.\end{aligned}$$

Note that $dR_\theta(L_1)=d\tilde{\phi}_\tau(L_2)$. For any $v\in\check{Z}$ such that $dJ(v)=0$ and any $w_2\in\check{Z}^\perp$, the condition that $v+w_2\in L_2$ implies that $v\in T^h_{p_2}P$ and that the image of w_2 under the identification used in Lemma 2.8 is v. The further condition $d\tilde{H}(v)=0$ restricts the image of w_2 to the energy surface. The same reasoning applies to the relation between $s\Xi_h+d\tilde{\phi}_t(v)$ and w_1.

□

There is a natural volume form on U, obtained by combining the Liouville form on the symplectic normal to $\mathbb{R}\Xi$ with ds and the canonical volume form on $\mathbb{R}\Xi$.

Lemma 2.12 *The image of product of the canonical half-forms on $\check{\Gamma}$ and $\check{\Sigma}$ under the map (53) is the product of the canonical half-forms on $\check{\Gamma}\circ\check{\Sigma}$ and U.*

PROOF As explained above, this map comes from the exact sequence

$$(63)\qquad 0\to\check{\Gamma}\circ\check{\Sigma}\to\check{\Gamma}\oplus\check{\Sigma}\to(U_1^\perp/U_1)\oplus U\to U\to 0.$$

Note that all of the spaces in the sequence have canonical half-forms. The computation simply amounts to writing down metabases corresponding to these half-forms and taking the square root determinant of the metaplectic transformation that relates them. The calculation is greatly simplified by dividing the problem up into three independent sectors:

the symplectic normal to $\mathbb{R}\Xi$, the space spanned by ∂_r and ∂_Θ, and the space spanned by Ξ and its symplectic partner. The second and third sectors are easily seen to contribute factors of 1. The first sector contributes the square root of the restriction of ϕ_{T*} to the symplectic normal to the flow. Since this metaplectic map is defined by continuity from the identity, the square root is 1.

□

Proposition 2.13 *The Kostant pairing of $e_{p_1} \otimes \overline{e_{p_2}}$ with the natural half-form on U yields*

$$\langle M_\tau^{-1} e_{p_1}, P_\Xi(e_{p_2}) \rangle. \tag{64}$$

Proof Choose a symplectic basis for the symplectic normal to $\mathbb{R}\Xi$ in $T^h_{p_2}P$, and fix a vector $\partial_\perp$ for which $\omega(\Xi, \partial_\perp) = 1$ and which is orthogonal to the rest of $T^h_{p_2}H^{-1}(E)$. This gives the identifications

$$T^h_{p_2}P \cong \mathbb{R}^{2n-2} \oplus \mathbb{R}^2, \qquad H_\infty(T^h_{p_2}P) \cong \mathcal{S}(\mathbb{R}^{n-1} \oplus \mathbb{R}). \tag{65}$$

With the $d\phi_\tau$ and the metaplectic quantization M_τ, we can make corresponding identifications of $T^h_{p_1}P$ and $H_\infty(T^h_{p_1}P)$.

Choose coordinates so that

$$T^h_{p_1}P \oplus T^h_{p_2}P \cong \{(x_1, y_1, w_1, z_1; x_1, y_1, w_1, z_1); x_j, y_j \in \mathbb{R}^{n-1}, w_j, z_j \in \mathbb{R}\}, \tag{66}$$

with the obvious symplectic pairing. Then $e_{p_1} \otimes \overline{e_{p_2}}$ becomes a function $f(x_1, w_1)\overline{g(x_2, w_2)}$, and

$$U = \{(x, y, w_1, 0; x, y, w_2, 0)\}, \tag{67}$$

(note that w is the coordinate corresponding to Ξ). According to Kostant's theorem, the element $h(x_1, w_1; x_2, w_2)$ of $\mathcal{S}'(\mathbb{R}^{n-1} \oplus \mathbb{R})$ associated to U is the solution of

$$(x_1 - x_2)h = 0, \qquad (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})h = 0, \qquad w_1 h = 0, \qquad w_2 h = 0, \tag{68}$$

namely,

$$h(x_1, w_1; x_2, w_2) = \delta(x_2 - x_1)\delta(w_1)\delta(w_2). \tag{69}$$

Thus the pairing gives

$$\int f(x, 0)\overline{g(x, 0)}\, d^{2n-2}x. \tag{70}$$

To write this pairing in an invariant way, note that

$$\int f(x, 0)\overline{g(x, 0)}\, d^{2n-2}x = \int\int_{-\infty}^{\infty} f(x, w)e^{itw}g(x, w)\, d^{2n-2}x\, dw\, dt. \tag{71}$$

Also note that the action of Ξ on $H_\infty(T^h_{p_2}P)$ in the Heisenberg representation is $\Xi \cdot f(x, w) = wf(x, w)$. Thus (70) is the inner product claimed.

□

Now, combining Propositions 2.9 and 2.10 with Lemma 2.12 and Proposition 2.13, we obtain Theorem 2.7.

3 Co-adjoint Orbits.

In this section we discuss the case when X is an integral co-adjoint orbit of a compact semisimple Lie group, G. We will show that on the symmetric orbits (defined by the condition (87)) one can define a global sub-principal symbol for Toeplitz operators, in the sense of [8], Chapter 11 (see (50) and the discussion around it). Moreover, for the symmetric elements of the universal enveloping algebra of $\mathfrak{g}$ (i.e. those that are symmetric products of right-invariant vector fields) this subprincipal symbol is equal to zero. For such operators the conclusion of Theorem 1.1 simplifies considerably.

3.1 Preliminaries

Let T be a maximal torus in the compact connected Lie group G, and let $\mathfrak{t} \subset \mathfrak{g}$ be its Lie algebra. Choosing an Ad-invariant Euclidean inner product on $\mathfrak{g}$, $\langle , \rangle$, we will identify adjoint and co-adjoint orbits. Let us fix an integral element $\phi \in \mathfrak{t}$, and let X denote the (co)adjoint orbit through X. We need to briefly recall how the pre-quantum line bundle of X and its Kähler polarization are defined.

Let $\mathfrak{g}_\phi$ the Lie algebra of the isotropy subgroup of ϕ, G_ϕ:

(72)
$$\mathfrak{g}_\phi \;=\; \{\, A \in \mathfrak{g}\,;\, [A\,,\,\phi] = 0\,\}\,.$$

Since $\forall\; A,\, B \in \mathfrak{g}_\phi \quad \langle \phi\,, [A, B]\,\rangle \;=\; -\langle [A, \phi]\,,\, B\rangle \;=\; 0$, the mapping

(73)
$$A \mapsto i\langle \phi\,,\, A\rangle$$

is an infinitesimal character of $\mathfrak{g}_\phi$. The integrality assumption on ϕ means precisely that this infinitesimal character exponentiates to a character

(74)
$$\chi\,:\, G_\phi \to S^1\,.$$

Define next

(75)
$$H \;:=\; \ker(\chi)\,,$$

and let $\mathfrak{h}$ denote its Lie algebra. Therefore

(76)
$$\mathfrak{h} \;=\; \{\, A \in \mathfrak{g}\,;\, [A, \phi] = 0 \text{ and } \langle A\,,\,\phi\rangle \;=\; 0\,\}\,.$$

The pre-quantum line bundle of X is then the homogeneous space

(77)
$$P \;:=\; G/H\,.$$

Lemma 3.1 *1. P has a natural structure of S^1 bundle over $X \cong G/G_\phi$. The S^1 action is induced by the right action of $\exp(t\phi)$ on G, which descends to P.*

2. Consider ϕ as an element in the dual $\mathfrak{g}^*$ of the Lie algebra of G. Then the left-invariant form defined by ϕ descends to a one-form on P, which is a connection form. Its curvature is the canonical symplectic structure on X.

Next we discuss the complex structure on X. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$, and consider the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} L_\alpha \,. \tag{78}$$

Here R is the set of real infinitesimal roots $\alpha : \mathfrak{t} \to \mathbb{R}$, $\alpha \neq 0$, and as usual

$$L_\alpha = \{ A \in \mathfrak{g}_{\mathbb{C}} \,;\, \forall B \in \mathfrak{t}_{\mathbb{C}} \quad [B, A] = i\alpha(B)A \} \,. \tag{79}$$

Consider next the subsets of R defined according to the sign of the roots on ϕ

$$R^+_\phi = \{ \alpha \in R \,;\, \alpha(\phi) > 0 \} \,,\; R^-_\phi = \{ \alpha \in R \,;\, \alpha(\phi) < 0 \} \,, \tag{80}$$

and let

$$\mathcal{N}^\pm_\phi = \bigoplus_{\alpha \in R^\pm_\phi} L_\alpha \,. \tag{81}$$

Observe that

$$\mathfrak{g}_\phi = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R \,;\, \alpha(\phi) = 0} L_\alpha \,. \tag{82}$$

The complex structure on X is induced by this decomposition of $\mathfrak{g}_{\mathbb{C}}$, viz.

$$\mathfrak{g}_{\mathbb{C}} = \mathcal{N}^+_\phi \oplus \mathcal{N}^-_\phi \oplus \mathfrak{g}_\phi \,. \tag{83}$$

The following is known.

Lemma 3.2 *The two $\mathcal{N}$'s are Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$.*

It follows that

$$T_\phi X \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}} / (\mathfrak{g}_\phi \otimes \mathbb{C}) = \mathcal{N}^+_\phi \oplus \mathcal{N}^-_\phi \,. \tag{84}$$

Proposition 3.3 *The adjoint action of G_ϕ preserves the decomposition (83), and therefore the decomposition (84) extends to a G-invariant almost complex structure on X. This structure is in fact integrable.*

The integrability follows from the fact that the $\mathcal{N}$'s are Lie subalgebras.

The complex vector space $\mathcal{N}^+_\phi$ has a natural Hermitian structure, namely

$$h(A, B) = i \, \langle \phi \,,\, [A, \overline{B}] \rangle \,. \tag{85}$$

(Here the pairing $\langle , \rangle$ is extended to $\mathfrak{g}_{\mathbb{C}}$ complex-bilinearly.) The imaginary part of the Hermitian structure is precisely the natural symplectic structure on X, induced by the bilinear form on $\mathfrak{g}$

$$(A, B) \mapsto \langle \phi \,,\, [A, B] \rangle \,. \tag{86}$$

We are identifying $\mathfrak{t}$ with its dual using the inner product, and therefore we consider the roots as elements in $\mathfrak{t}$.

Lemma 3.4 *The action of G_ϕ on $\mathcal{N}_\phi^+$ is unitary. The action of H is a special unitary action if ϕ is proportional to the sum of the ϕ-positive roots,*

$$\rho_\phi := \sum_{\alpha \in R_\phi^+} \alpha \, .$$

Proof The unitarity of G_ϕ on $\mathcal{N}_\phi^+$ is trivial to verify. The infinitesimal character of this action is precisely ρ_ϕ (T is a maximal torus of G_ϕ), so this character vanishes on $\mathfrak{h} \cap \mathfrak{t}$ if ϕ is proportional to ρ_ϕ.

□

For definiteness, we'll take henceforth

$$\phi = \rho_\phi \, . \tag{87}$$

Although we won't need it, we mention that this means precisely that the orbit X through through ϕ is a symmetric space of G. From our point of view the advantage of orbits of this type is that on them one can globally define the subprincipal symbol of any Toeplitz operator, as discussed in the third remark after Theorem (2.7).

Lemma 3.5 *Let X be the coadjoint orbit through (87), and $P \to X$ its prequantum circle bundle. Let $Z \subset T^*P$ be as in §2. Then there is a left-invariant section, e, of the bundle $H_\infty(Z^\perp)$ such that $\forall \sigma \in Z$ the spinor part of the symbol of the Szego projector at σ is $e_\sigma \otimes \overline{e_\sigma}$.*

Proof Let $o \in P = G/H$ denote the coset of H and denote the connection form on P by α. As we saw in §2, one can identify by the natural cotangent projection $Z^\perp_{(o,\alpha_0)}$ with the maximal complex subspace in T_oP, which in turn is $\mathcal{N}_\phi^+$. Choose e_o as the ground state of the harmonic oscillator, with an arbitrary phase. For concreteness, let us realize the metaplectic representation of $Z^\perp_{(o,\alpha_0)}$ in the Bargmann space of $\mathcal{N}_\phi^+$, and choose e_o to be the constant function one. The obstruction to being able to extend e_o to a left-invariant section is whether the action of H_ϕ on $\mathcal{N}_\phi^+$, followed by the metaplectic representation, leaves e_o invariant. But it is well-known that in the Bargmann space realization of the metaplectic representation, *special* unitary maps act by translations, and therefore they leave invariant the ground state of the harmonic oscillator. Hence, by lemma (3.4), e_o *is* invariant under H_ϕ.

□

Following [8] (page 85), we now define the sub-principal symbol of Toeplitz operators in the Hardy space of P (see also (50)):

Definition 3.6 *Keeping the assumptions and notation of Lemma 3.5, let Q be a classical pseudodifferential operator on P that commutes with the Szegö projector, Π, and let $T = \Pi Q \Pi$ the Toepltiz operator on P with multiplier Q. Let the function c be defined as in the hypotheses of Theorem 2.7, using the section e of Lemma 3.5 (see also [8], page 85; our c is by definition $-ic(Q)$ of formula (11.6) in* op.cit.*). Then we define the Toeplitz subprincipal symbol of T to be the function on Z*

$$\sigma_{sub}[Q] + c \, ,$$

where $\sigma_{sub}[Q]$ is the subprincipal symbol of the pseudodifferential operator Q restricted to Z.

We will need the following.

Lemma 3.7 *The subprincipal symbol of the Toeplitz operator with multiplier $D_\theta = -i\partial/\partial\theta$, where $\partial/\partial\theta$ is the infinitesimal generator of the S^1 action, is constant and equal to $\frac{1}{2}\|\rho_\phi\|^2$.*

PROOF Since D_θ is of order one, the subprincipal symbol of the associated Toeplitz operator is a function on Z homogeneous of degree zero. Since that Toeplitz operator commutes with the G action, the function is necessarily a constant. To evaluate it, observe that the pseudodifferential subprincipal symbol of D_θ is zero, and therefore the Toeplitz subprincipal symbol is just the derivative of the section e with respect to the S^1 action on P. This is computed by considering the metaplectic quantization of the adjoint action of $\exp(t\phi)$ on $\mathcal{N}^+_\phi$, as follows. Recall that the S^1 action on $P = G/G_\phi$ is induced by the right action of $\exp(t\phi)$. Therefore the derivative of the section e is minus the metaplectic quantization of $\mathrm{ad}_\phi|_{\mathcal{N}^+}$ acting on e. But, since the action of G_ϕ on $\mathcal{N}^+$ is unitary and e is the ground state of the Harmonic oscillator, it is known that the result of this action is simply

(88)
$$-\frac{1}{2}\mathrm{Tr}\left(\mathrm{ad}_\phi|_{\mathcal{N}^+}\right),$$

which in this case is $-1/2 < \phi, \rho_\phi >$ (for example, this follows directly from Proposition 4.39 in [14]). Recalling that we have taken $\phi = \rho_\phi$ we obtain the result.

□

3.2 Theorem 1.1 for operators in the enveloping algebra.

We will now consider operators arising from the universal enveloping algebra $\mathfrak{u}$ of G, which we think of as the algebra of right-invariant differential operators on G. We recall (see for example [12] Ch. 2) that if $Q \in \mathfrak{u}$ then we can uniquely write it in the form

(89)
$$Q = Q_s + Q_1$$

where Q_s is a sum of *symmetric* products of right-invariant vector fields, each of the same degree as Q, and the degree of Q_1 equals degree$(Q) - 1$. By right invariance, the operators in $\mathfrak{u}$ descend to operators on $P = G/H$ and they clearly form an algebra. Let $\mathfrak{u}_P$ denote this algebra of operators on P. Alternatively, $\mathfrak{u}_P$ is the algebra of operators generated by the vector fields on P induced by the (left) action of G on P. Observe that, since the S^1 action on P is induced by the right-action of a one-parameter subgroup in G_ϕ, every operator in $\mathfrak{u}_P$ commutes with the S^1 action (by right-invariance of the elements of $\mathfrak{u}$). Moreover, since the action of G on P commutes with the Szegö projector every operator in $\mathfrak{u}_P$ also commutes with with the Szegö projector.

We next discuss symbolic matters. By right-invariance, the symbols of the elements of $\mathfrak{u}$ (considered as differential operators on G) are determined by their restriction to $T^*_eG = \mathfrak{g}^*$. The map

(90)
$$\sigma : \mathfrak{u} \to C^\infty(\mathfrak{g}^*)$$

has for image the complex-valued polynomial functions on $\mathfrak{g}^*$. On the other hand, given $Q \in \mathfrak{u}$, the corresponding operator $Q_P \in \mathfrak{u}_P$ has a principal symbol (as a differential operator on P) which is a function on T^*P. It is homogeneous, and by S^1 invariance, its restriction to $Z \cap \{J = 1\} \subset T^*P$ descends to a function on $X \subset \mathfrak{g}^*$. (Recall that $J : T^*P \to \mathbb{R}$ is the Hamiltonian generating the S^1 action).

Lemma 3.8 *For every $Q \in \mathfrak{u}$, the function induced on X by the principal symbol of Q_P is simply the restriction to X of $\sigma(Q)$, where σ is the map (90).*

PROOF Let $\mathcal{P} \subset T^*G$, be the annihilator of the vectors tangent to the fibers of the fibration $G \to P$. Notice that $\mathcal{P}$ is right-invariant under the action of H, and that there is a natural projection

$$\pi : \mathcal{P} \to T^*P\,. \tag{91}$$

For every $Q \in \mathfrak{u}$, the principal symbol of the corresponding operator in $\mathfrak{u}_P$ is the function on T^*P which, when pulled-back by (91), agrees with the restriction of the principal symbol of Q to $\mathcal{P}$. On the other hand, we claim that

$$\pi^{-1}(Z) \;=\; \{\,(g, \mathrm{Ad}^*_g(r\phi))\,;\, g \in G\,,\, r > 0\,\}\,, \tag{92}$$

where we are identifying T^*G with $G \times \mathfrak{g}^*$ using right translations. Indeed by left-invariance of Z and left-equivariance of π under G it is enough to verify (92) at the identity, which follows from the discussion around Lemma 3.5. The lemma follows easily from (92) and the definition of σ.

□

Proposition 3.9 *For every $Q \in \mathfrak{u}$, the subprincipal symbol of the Toeplitz operator defined by Q_P in the Hardy space of P is the function, q_1, on Z induced by the principal symbol of Q_1.*

PROOF By the equivariance of the section e, the subprincipal symbol of the Toeplitz operators defined by the vector fields induced by the action of G are zero. Using the symbol calculus for subprincipal symbols (Proposition 11.9 in [8]), it follows easily by induction that the subprincipal symbol of the Toeplitz operator induced by Q_s is zero.

□

Let $Q \in \mathfrak{u}_P$; we wish to normalize Q so that it has order one. For this purpose we recall a construction found for example in [25]. Let Δ denote the Laplacian on G associated to the bi-invariant metric. Then Δ is in the center of $\mathfrak{u}$, and recall that

$$\pi_\lambda(\Delta + \|\rho\|^2) \;=\; \|\lambda + \rho\|^2\,,$$

where ρ is half the sum of the positive roots and π_λ the representation with highest weight λ. Using a suitable function of Δ, one can easily construct an operator A_1 with the following properties:

Lemma 3.10 *There exists a first-order, invariant, self-adjoint elliptic pseudodifferential operator A_1 on P that commutes with the Szego projector and with every operator in $\mathfrak{u}_P$ and that defines the same Toeplitz operator as D_θ. In particular, the Toepliz subprincipal symbol of A_1 is $\frac{1}{2}\|\rho\|^2$.*

We now introduce the following notation. Define

$$A \;:=\; A_1 - \frac{1}{2}\,\|\rho_\phi\|^2\,. \tag{93}$$

Observe that the Toeplitz subprincipal symbol of A is zero. Next, for every $Q \in \mathfrak{u}_P$ of degree m and every $k = 1, 2, \ldots$, let

$$T_Q^k = \Pi_k \circ Q \circ A^{1-m} \circ \Pi_k \, . \tag{94}$$

This is a first-order Toeplitz operator with same eigenfunctions as those of $\Pi_k Q \Pi_k$. Moreover, if

$$\lambda_1^{(k)} \leq \cdots \leq \lambda_{d_k}^{(k)}$$

denote the eigenvalues of $\Pi_k Q \Pi_k$, with multiplicities, then the eigenvalues of T_Q^k are simply

$$E_j^k = \frac{\lambda_j}{(k - \|\rho_\phi\|^2/2)^{m-1}} \quad j = 1, \ldots, d_k \, . \tag{95}$$

Theorem 3.11 *Let ψ_j^k denote an eigenfunction of $\Pi_k Q \Pi_k$ with eigenvalue $\lambda_j^{(k)}$. Then the conclusions of Theorems 1.1 and 1.2 remain valid in the present context. Moreover, in the formula for $C_{\tau,0}$ of Theorem 1.2 one has $c = 0$ and $\sigma_{sub}[Q] = q_1$, the function of Proposition 3.9. In particular, $q_1 = 0$ if Q is symmetric, i.e. if Q_1 as defined by (89) is equal to zero.*

PROOF The proof is identical to that of Theorems 1.1 and 1.2., applied to the Toeplitz operator $T_Q = \Pi \circ Q \circ A^{1-m} \circ \Pi$ where Π is the Szego projector of Π. As for the calculation of the Toeplitz subprincipal symbol of T_Q, observe that the Toeplitz subprincipal symbol of A is zero and apply Proposition 3.9.

□

4 The semi-classical trace formula

4.1 Statement of the theorem

The techniques of §2 also serve to prove the semi-classical trace formula for the operator S_k. Given the setup of §1, let E be a regular value of $H : X \to \mathbb{R}$. The Hamilton flow ϕ_t of H is said to be clean on $H^{-1}(E)$ iff the immersions f, Δ

$$\begin{array}{ccc} & & H^{-1}(E) \times \mathbb{R} \\ & & \downarrow f \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{96}$$

where $f(x,t) = (x, \phi_t(x))$ and $\Delta(x) = (x,x)$ intersect cleanly. Recall that this in turn means that each connected component of the set

$$\mathcal{Y} = \{ (x, \tau) \in H^{-1}(E) \times \mathbb{R} \, ; \, \phi_\tau(x) = x \} \tag{97}$$

is a submanifold of $H^{-1}(E) \times \mathbb{R}$ and at every point $(x, \tau) \in \mathcal{Y}$ one has

$$T_{(x,\tau)}\mathcal{Y} = \{ (v, \delta\tau) \in T_x H^{-1}(E) \times \mathbb{R} \, ; \, d(\phi_\tau)_x(v) + \delta\tau \, \Xi_x = v \} \, . \tag{98}$$

We will henceforth assume this is the case. To state the trace formula we need to recall a known result:

Lemma 4.1 *The holonomy function*

$$\begin{array}{rcc} \mathfrak{h} : \mathcal{Y} & \longrightarrow & S^1 \\ (x, \tau) & \mapsto & \exp(i \int_0^\tau \alpha) \end{array}$$

where α is defined by 1 and the integral is along the trajectory of ϕ with between x and $\phi_\tau(x)$, is locally constant.

Let

$$\mathcal{Y} = \bigcup_{j \in J} \mathcal{Y}_j \tag{99}$$

denote the decomposition of $\mathcal{Y}$ into its connected components. $\forall j \in J$ denote by $\mathfrak{h}_j$ the holonomy of the periodic trajectories in $\mathcal{Y}_j$ and by d_j the dimension of $\mathcal{Y}_j$. We index the components above so that

$$\mathcal{Y}_0 = \{ (x, 0) \, ; \, x \in H^{-1}(E) \} . \tag{100}$$

We can now state the trace formula:

Theorem 4.2 *Let A be a zeroth order pseudodifferential operator on P commuting with D_θ. Then, under the "cleanness" assumption above, $\forall \varphi$ Schwartz function on the line with $\hat{\varphi}$ compactly supported we have the asymptotic expansion*

$$\sum_{j=0}^{\infty} \langle A\psi_j^k \, , \, \psi_j^k \rangle \, \varphi(k(E_j^k - E)) \; \sim \; \sum_{j \in J} \sum_{l=0}^{\infty} C_{j,l}(a\hat{\varphi}) \, (\mathfrak{h}_j)^k \, k^{(d_j - 1)/2 - l}, \tag{101}$$

where a is the function on X induced by the principal symbol of A and $C_{j,0}(a\hat{\varphi})$ can be expressed as an integral

$$C_{j,0}(\hat{\varphi}) = \int_{\mathcal{Y}_j} a\hat{\varphi} \, d\nu_j \tag{102}$$

where $d\nu_j$ is a natural density on $\mathcal{Y}_j$ (more about it later). In particular, the leading term is $O(k^{n-1})$, with coefficient given by $C_{0,0} = (2\pi)^{-n} \, \hat{\varphi}(0) \int_{H^{-1}(E)} a \, d\ell$ where $d\ell$ denotes Liouville measure.

4.2 The Proof

The proof follows the general outline of the proof of Theorem 1.1. To lighten up the notation we'll first take $A = I$, and then observe that the proof generalizes to A's as in the statement of the Theorem. Once again take $B = e^{-itQ}\Pi$, but now choose the Lagrangian FIO F to be

$$F : f(t, p, q) \mapsto \int e^{i\kappa(s + tE - \theta)} \hat{\varphi}(t) f(t, R_\theta(p), p) d\kappa \, d\theta \, dt \, dp. \tag{103}$$

Then $\Upsilon = F(\mathcal{B})$ is

$$\Upsilon(s) = \sum_{k=0} \sum_{j=0}^{\infty} \phi(k(E_j^k - E)) e^{iks}, \tag{104}$$

yielding the asymptotics we need.

The Schwartz kernel of F is a distribution

$$\mathcal{K}_F \in I^{n+1/2}(S^1 \times \mathbb{R} \times P \times P, \Gamma), \tag{105}$$

where the Lagrangian relation is

$$\Gamma = \{(\theta - tE, J(m); t, EJ(m); R_\theta(m); m); \theta \in \mathbb{R}/2\pi\mathbb{Z}, t \in \mathbb{R}, m \in T^*P\}. \tag{106}$$

Lemma 4.3 *The assumption made above (on the cleanness of the Hamilton flow of H on $H^{-1}(E)$) guarantees that the composition of Γ with Σ, the isotropic relation of B, is clean.*

Proof Recall that the isotropic relation of B is the moment Lagrangian (24),

$$\Sigma = \{(t, \tilde{H}(z); \tilde{\phi}_t(z); z);\ z \in Z, t \in \mathbb{R}\} \subset T^*\mathbb{R} \times Z^- \times Z.$$

By definition the composition of Γ with Σ is clean iff the pull-back diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & T^*(\mathbb{R} \times P \times P) \end{array} \tag{107}$$

is clean. We must prove this is the case assuming that (96) is clean. The elements of the fiber product, $\mathcal{F}$, are precisely those elements of

$$T^*(S^1 \times \mathbb{R} \times P \times P) \times T^*(\mathbb{R} \times P \times P)$$

of the form

$$((\theta - tE, J(z)\,;\, t, EJ(z)\,;\, R_\theta(z)\,;\, z)\,;\ (t, \tilde{H}(z)\,;\, \tilde{\phi}_t(z)\,;\, z\,)) \tag{108}$$

where $z \in Z$ and

$$\left\{ \begin{array}{rcl} R_\theta(z) & = & \tilde{\phi}_t(z); \\ \tilde{H}(z) & = & EJ(z). \end{array} \right. \tag{109}$$

We first must prove that $\mathcal{F}$ is a submanifold. Introduce the space

$$\mathcal{Y}_P := \{\, (p,t) \in \mathbb{R} \times P\ ;\ (\pi(p), t) \in \mathcal{Y}\,\}\,. \tag{110}$$

This is a submanifold of $\mathbb{R} \times P$; indeed it is the pull-back of P to $\mathcal{Y}$ under the natural projection $\mathcal{Y} \to X$. Now observe that there is a natural diffeomorphism

$$f\ :\ \mathbb{R}^+ \times \mathcal{Y}_P \longrightarrow \mathcal{F}\,. \tag{111}$$

(More precisely, f is an immersion into $\Gamma \times \Sigma$ whose image is $\mathcal{F}$.) Explicitly, f maps (r, p, t) to the point (108) where

$$z = r\alpha_p$$

and θ is defined by the first condition (109), which indeed has a unique solution modulo 2π by the periodicity of $\pi(p)$. The second condition (109) follows from the fact that $\pi(p) \in H^{-1}(E)$. The verification of the tangent space condition will be left to the reader.

□

By the invariance of Hermite distributions under clean composition with FIO's, we can conclude that

$$\Upsilon \in \bigoplus_{j\in J_\varphi} I^{d_j/2}(S^1, \{\mathfrak{h}_j\}\times\mathbb{R}_+)\,, \tag{112}$$

where

$$J_\varphi := \{\, j \in J\ ;\ \exists (x,t)\in \mathcal{Y}_j \text{ s.t. } t\in \operatorname{supp}(\hat\varphi)\,\} \tag{113}$$

is a finite set (by the compactness of the support of $\hat\varphi$). This proves the existence of the asymptotic expansion (101), and the leading coefficients of the expansion are obtained from the symbol of Υ.

We now turn to the calculation of the symbol of Υ at a particular singular point. Let us fix henceforth a point (108) in $\mathcal{F}$. As in §2, we will denote the tangent space to a manifold Y (at an obvious point) by $\check{Y}$.

The Appendix outlines the Hermite symbol calculus. Note that the map defined there yields the product of the symbol of Υ with a density on the vector space U_0, which is by definition is the tangent space to the fiber of the projection from $\mathcal{F}\to T^*S^1$. The symbol computation is completed by integrating this density over the (compact) fiber of this projection.

The key fact that reduces the calculation to the standard trace-formula calculation on X is that $\check\Gamma$ splits

$$\check\Gamma = \check\Gamma_1 \oplus \check\Gamma_2 \tag{114}$$

where

$$\check\Gamma_1 \subset T(T^*(S^1\times\mathbb{R}))\times \check Z\times\check Z \quad\text{and}\quad \check\Gamma_2\subset\{0\}\times\{0\}\times\check Z^\perp\times\check Z^\perp\,. \tag{115}$$

$\check\Gamma_2$ is just the diagonal and carries a natural half-form induced from the symplectic form on $\check Z^\perp$. Both $\check\Gamma_j$ are Lagrangian subspaces of the symplectic vector spaces on the right-hand side of the inclusions in (115). Moreover, thinking of the $\check\Gamma_j$ as relations from $T(T^*(S^1\times\mathbb{R}\times P\times P))$ to $T(T^*S^1)$, the domain of $\check\Gamma_2$ does not intersect $\check\Sigma$, so $\check\Gamma_2$ will not play a significant role in the calculation. The details of this are presented in the Appendix.

The symbol of $\mathcal{B}$ is given in Proposition 2.9, and the symbol of F is just $\hat\varphi$ times $\sqrt{ds}\otimes\sqrt{dt}\otimes$ (the canonical half form on T^*P). We wish to apply Proposition A.5 to the symbol map for $F(\mathcal{B})$, where we take $\mathcal{Z} = T(T^*\mathbb{R})\times\check Z\times\check Z$. In this case $\Sigma^\perp/\Sigma = \mathcal{Z}^\perp = \check Z^\perp\times\check Z^\perp$, and we take

$$\beta = \exp\Big\{-i\int_0^\tau(\sigma_{sub}[Q]+c)dt\Big\}\; e\otimes\overline{e} \quad \in H_\infty(\check Z^\perp)\otimes H_\infty(\check Z^\perp). \tag{116}$$

For μ we have the natural half form on $\check\Gamma_2$ mentioned above, which means the Kostant pairing amounts to nothing more than $\langle e,e\rangle_{H(\check Z^\perp)} = 1$, so that

$$\langle\beta,\mu\rangle_K = \exp\Big\{-i\int_0^\tau(\sigma_{sub}[Q]+c)dt\Big\}. \tag{117}$$

So by Proposition A.5, we now have only to calculate the result of the Lagrangian symbol map λ. But now we can proceed as in the trace formula calculation in the presence of circular symmetry in [18]. Namely, we move the calculation down to X by considering symplectic reduction under the S^1 action. X is the symplectic reduction of Z (the quotient of $Z\cap J^{-1}(1)$ by S^1), and the flow of H is the reduced flow of $\tilde H$. Under these reductions the space U_0 is identified the tangent space to the fixed point component $\mathcal{Y}_j$, resulting in the formula

(118) $$C_{0,j}(\hat{\varphi}) = \int_{\mathcal{Y}_j} \hat{\varphi}(t)\, d\nu_j,$$

where

(119) $$d\nu_j = e^{-i\int_0^\tau (\sigma_{sub}[Q]+c)dt} d\mu_t,$$

and the measures $d\mu_t$ are those appearing in Theorem 5.6 of [18]. As the construction of the $d\mu_t$ was described in detail in this paper, we will not repeat these details here.

Instead we consider two important cases. First, the contribution from $\mathcal{Y}_0$ is simply

(120) $$C_{0,0}(\hat{\varphi}) = (2\pi)^{-n}\hat{\varphi}(0)\,\mathrm{vol}(H^{-1}(E)).$$

For the other case, suppose that $\mathcal{Y}_j = (\gamma, \tau)$, where γ is an isolated, closed trajectory of ϕ. Then, following [18], the contribution is

$$C_{0,j}(\hat{\varphi}) = \frac{1}{2\pi}\hat{\varphi}(\tau) e^{-i\int_0^\tau (\sigma_{sub}[Q]+c)dt} \frac{\tau_0}{\det(I-P_\gamma)^{1/2}},$$

where τ_0 is the primitive period of γ, and P_γ is the Poincaré map associated to γ.

To obtain the result with an observable A as in the statement of the Theorem, repeat the above working with

$$B_A \;=\; \tilde{A}\, e^{-itQ}\,\Pi$$

where $\tilde{A}$ is a pseudodifferential operator commuting with Π and with D_θ and defining the same Toeplitz operator as A. B_A has the same microlocal properties as B, except that its symbol is the symbol of B multiplied by the symbol of A.

4.3 Weyl-type estimates

In this final section we remark that the Tauberian lemmas used in [21] and [22] also apply in the situation of this paper. We obtain immediately:

Theorem 4.4 *If E is a regular value of H and the set of periodic points in $H^{-1}(E)$ has Liouville measure equal to zero,*

(121) $$\sharp\{E_j^k\, ;\, \mid k(E_j^k - E) \mid \le c\} \;=\; \frac{2c}{(2\pi)^n}\,\mathrm{Vol}(H^{-1}(E))k^{n-1} + o(k^{n-1})\,.$$

Theorem 4.5 *If $\pi(p)$ is either non-periodic or belongs to an unstable periodic trajectory of period T_γ and action $S_\gamma \in 2\pi\mathbb{Z}$, then*

(122) $$\sum_{|k(E_j^k - E)| \le c} \mid \psi_j^k(p) \mid^2 \;=\; 2c \times C_{0,0} k^{n-1/2} + o(k^{n-1/2}) \qquad \textit{if } \pi(p) \textit{ is not periodic}$$

and

(123) $$\sum_{|k(E_j^k - E)| \le c} \mid \psi_j^k(p) \mid^2 \;=\; \sum_l C_{lT_\gamma,0}\frac{2\sin(lT_\gamma c)}{lT_\gamma} k^{n-1/2} + o(k^{n-1/2}) \qquad \textit{if } \pi(p) \textit{ is periodic}$$

where $C_{\tau,j}$ is as in Theorem 1.1.

The case where $\pi(p)$ belongs to a stable periodic orbit can be treated as in [23].

A Appendix

A.1 The calculus of Hermite symbols

We begin with a review of the symplectic linear algebra that is at the core of the calculus of Hermite symbols. We begin by recalling the contents of §6 of [8], where the reader can find more details. Let V and W be symplectic vector spaces, $\Gamma \subset V \times W^-$ a Lagrangian subspace and $\Sigma \subset W$ an isotropic subspace. We think of Γ as a canonical relation from W to V; $\Gamma \circ \Sigma$ is an isotropic subspace of V. We assume given a symplectic spinor on Σ and a half-form on Γ. Recall that if $H_\infty(V)$ denotes the space of C^∞ vectors in the metaplectic representation of the metaplectic group of the symplectic vector space V, the space of symplectic spinors on Σ is $H_\infty(\Sigma^\perp/\Sigma) \otimes \bigwedge^{1/2}(\Sigma)$. The (linear) symbol map of the Hermite calculus is a linear map

$$(124)\quad H_\infty(\Sigma^\perp/\Sigma) \otimes \textstyle\bigwedge^{1/2}(\Sigma) \otimes \bigwedge^{1/2}(\Gamma) \to |\bigwedge|(U_0) \otimes H_\infty((\Gamma \circ \Sigma)^\perp/\Gamma \circ \Sigma) \otimes \bigwedge^{1/2}(\Gamma \circ \Sigma)$$

(The space U_0, defined below, is the tangent space to a compact manifold ; the final stage in the symbol calculation is to integrate over this manifold the density resulting from (124).)

Our first goal here is to describe the map (124). There are two ingredients in its construction, which will be examined separately. First however we must introduce some vector spaces.

Definition A.1

$$U_0 := \{ w \in \Sigma\,;\, (0,w) \in \Gamma \} \subset W\,,$$

$$U_1 := \{ w \in \Sigma^\perp\,;\, (0,w) \in \Gamma \} \subset W\,,$$

$$U := U_1/U_0 = \textit{image of } U_1 \textit{ in } \Sigma^\perp/\Sigma\,.$$

The first ingredient in the symbol map is a canonical isomorphism:

Lemma A.2 *There exists a canonical isomorphism*

$$\textstyle\bigwedge^{1/2}(\Sigma) \otimes \bigwedge^{1/2}(\Gamma) \cong \bigwedge^{1/2}(U_1^\perp) \otimes \bigwedge^{1/2}(U_0) \otimes \bigwedge^{1/2}(\Gamma \circ \Sigma)\,.$$

Proof Let

$$(125)\qquad \rho : \Gamma \oplus \Sigma \to U_1^\perp$$

be the map

$$\rho((v,w),w_1) = w - w_1\,.$$

One can show that the image of this map is exactly $U_1^\perp$. In fact one has two exact sequences:

$$(126)\qquad \begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & U_0 & & & & \\ & & \downarrow & & & & \\ 0 \to & & \ker(\rho) & \to \Gamma \oplus \Sigma \to & U_1^\perp & \to & 0 \\ & & \downarrow & & & & \\ & & \Gamma \circ \Sigma & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

The horizontal sequence is just the natural short exact sequence associated to the surjection ρ. The first non-trivial map in the vertical sequence is

$$w \mapsto ((0,w),w)\,; \tag{127}$$

observing that $\ker(\rho) = \{((v,w),w) \in \Gamma \oplus \Sigma\,\}$ we see that the cokernel of (127) is naturally $\Gamma(\Sigma)$.

Having established the existence of these exact sequences the desired isomorphism follows from the behavior of the functor $\bigwedge^{1/2}$ when applied to short exact sequences and to direct sums.

□

Remark: In case Σ is actually Lagrangian, then $U_1 = U_0$ and if we use the fact that the symplectic form of W gives us an isomorphism

$$\textstyle\bigwedge^{1/2} U_0^{\perp} \;\cong\; \overline{\bigwedge}^{1/2} U_0\,,$$

the previous Lemma gives us an isomorphism

$$\lambda\,:\,\textstyle\bigwedge^{1/2}(\Sigma)\otimes\bigwedge^{1/2}(\Gamma) \;\overset{\cong}{\to}\; |\bigwedge|(U_0)\otimes\bigwedge^{1/2}(\Gamma\circ\Sigma)\,. \tag{128}$$

This *is* the symbol map in the Lagrangian case.

The second ingredient in the Hermite calculus is the following:

Lemma A.3 *There exists a canonical map*

$$\mathcal{S}(\Sigma^{\perp}/\Sigma) \to \overline{\textstyle\bigwedge}^{-1/2}(U_1)\otimes\overline{\textstyle\bigwedge}^{1/2}(U_0)\otimes\mathcal{S}((\Gamma\circ\Sigma)^{\perp}/\Gamma\circ\Sigma)\,.$$

Proof First we need the following fact:

Claim 1: The subspace

$$U \subset \Sigma^{\perp}/\Sigma$$

is isotropic, and there is a natural identification

$$U^{\perp}/U \cong (\Gamma\circ\Sigma)^{\perp}/\Gamma\circ\Sigma\,. \tag{129}$$

Next we need a generalization of a map defined by Kostant.

Claim 2: Let Y (in our case we will take $Y = \Sigma^{\perp}/\Sigma$) be a symplectic vector space, and $U \subset Y$ an isotropic subspace. Then there is a natural map

$$H_{\infty}(Y) \to \overline{\textstyle\bigwedge}^{-1/2}(U)\otimes H_{\infty}(U^{\perp}/U)\,. \tag{130}$$

The desired map follows from these two claims, if we recall that $U = U_1/U_0$.

□

To obtain the symbol map (124), tensor the maps from the lemmas and use the fact that the symplectic form on W defines a natural identification

$$\overline{\textstyle\bigwedge}^{-1/2}(U_1)\otimes\textstyle\bigwedge^{1/2}(U_1^{\perp}) \cong \mathbb{C}\,.$$

A.2 Reduction of the symbol map for the trace formula

We will now consider the abstract setting of the symbolic calculation in the proof of the trace formula. Let V, W, Γ, and Σ be vector spaces as above, and let $\mathcal{Z} \subset W$ be a symplectic vector space. We make the following additional assumptions:

1. Σ is a Lagrangian subspace of $\mathcal{Z}$.

2. Γ splits as a direct sum $\Gamma = \Gamma_1 \oplus \Gamma_2$, where
$$\Gamma_1 \subset V \times \mathcal{Z} \quad \text{and} \quad \Gamma_2 \subset \{0\} \times \mathcal{Z}^{\perp}.$$

3. Γ_1 is Lagrangian in $V \times \mathcal{Z}^{-}$, and $\Gamma_2 = \{0\} \times U$.

(In the application to the trace formula, we take $V = T(T^*S^1)$, $W = T(T^*(\mathbb{R} \times P \times P))$, and
$$\mathcal{Z} = T(T^*\mathbb{R}) \times \check{Z} \times \check{Z}\,,$$
where the right-hand side is written in the notation of §4, and all the tangent spaces are taken at the appropriate points. Is is easy to check that the listed assumptions hold for this case.)

We will need the following fact:

Lemma A.4 *The composition*
$$Z^{\perp} \hookrightarrow \Sigma^{\perp} \to \Sigma^{\perp}/\Sigma$$
is an isomorphism of symplectic vector spaces.

We will identify below $\Sigma^{\perp}/\Sigma$ with $Z^{\perp}$ in this fashion.

The assumptions listed above imply that
$$\Gamma \circ \Sigma = \Gamma_1 \circ \Sigma$$
is a Lagrangian subspace of V. It follows that the subspace $U \subset \Sigma^{\perp}\Sigma \cong Z^{\perp}$ is also Lagrangian.

Our goal in this subsection is to explain how the Hermite symbol map, with Σ isotropic $\subset W$ and Γ the relation from W to V, reduces to the Lagrangian symbol map with Σ Lagrangian $\subset \mathcal{Z}$ and Γ_1 the relation from $\mathcal{Z}$ to V.

To this end, fix $\mu \in \bigwedge^{1/2}\Gamma_2$ and an element $\beta \in H_{\infty}(\Sigma^{\perp}/\Sigma)$. Contracting the symbol map (124) with them, we obtain a map
$$f_{\mu,\beta} : \textstyle\bigwedge^{1/2}(\Sigma) \otimes \bigwedge^{1/2}(\Gamma_1) \;\to\; |\bigwedge|(U_0) \otimes \bigwedge^{1/2}(\Gamma \circ \Sigma)\,.$$

Observe that the assumptions above allow us to naturally identify

(131) $$\beta \in H_{\infty}(\mathcal{Z}^{\perp}) \quad \text{and} \quad \mu \in \textstyle\bigwedge^{1/2} U\,.$$

Since $U \subset \mathcal{Z}^{\perp}$ is Lagrangian, we can form the Kostant pairing of e and μ, which we will denote by $< \beta, \mu >_K$. With this notation the result we are after is the following:

Proposition A.5 *Let $\lambda : \bigwedge^{1/2}(\Sigma) \otimes \bigwedge^{1/2}(\Gamma_1) \;\to\; |\bigwedge|(U_0) \otimes \bigwedge^{1/2}(\Gamma \circ \Sigma)$ be the Lagrangian calculus map (defined as in (128)) associated to Z, Σ, and Γ_1. Then*
$$f_{\mu,\beta} = < \beta, \mu >_K \; \lambda\,.$$

The proof is simply to go over the definition of the map $f_{\mu,\beta}$, the main point being that the hypotheses on Γ,Z and Σ allow us to identify the Kostant pairing in the definition of $f_{\mu,\beta}$ with the pairing of e and μ.

B The harmonic oscillator

For the harmonic oscillator with n degrees of freedom the calculation of the asymptotics of the sums of the squares of the lengths of the eigenstates with a given eigenvalue (in the regime of Theorem 1.1) can be done explicitly. We present here the highlights of the calculation.

We begin by recalling the abstract setting of the harmonic oscillator. Let $\mathcal{H}$ be the Hilbert space of the theory, and a_j, a_j^*, $j = 1, .., n$ be n commuting couples of creation and annihilation operators on $\mathcal{H}$, that is a_j^* is the adjoint of a_j and

$$[a_j, a_j^*] \;=\; \hbar \tag{132}$$

with all other commutators equal to zero. Let

$$N \;=\; \sum_{j=1}^{n} a_j^*\, a_j\,.$$

Then the harmonic oscillator quantum hamiltonian is

$$S \;=\; N + \hbar/2\,.$$

We recall that the spectrum of N (without multiplicities) equals the set of eigenvalues $\{\, m\hbar\,;\; m = 0, 1, 2, \dots\}$. Zero is a simple eigenvalue; denote by $|0\rangle$ a corresponding normalized eigenstate. Then all other normalized eigenstates are obtained as follows. For every multi-index $\nu = (\nu_1, \,\dots\, \nu_n)$, let

$$|\nu\rangle \;=\; \frac{1}{\sqrt{\nu!\hbar^{|\nu|}}}\,(a_1^*)^{\alpha_1}\cdots(a_n^*)^{\alpha_n}\,|0\rangle \tag{133}$$

where $\nu! \;=\; \nu_1!\cdots\nu_n!$ and $|\nu| \;=\; \sum_{j=1}^n \nu_j$. Then $N|\nu\rangle \;=\; |\nu||\nu\rangle$. The eigenvalue $|\nu|$ is therefore degenerate, the degeneracy arising from multi-indices ν' with $|\nu'| \;=\; |\nu|$. The multiplicity of $E = |\nu|$ can be shown to be equal to the binomial coefficient $C_{|\nu|}^{n+|\nu|-1}$.

Next recall that the theory above can be realized in the context of this paper, that is, with $\mathcal{H}$ a space of holomorphic sections of a line bundle and with S a Toeplitz operator. The underlying phase space is $X \;=\; \mathbb{C}^n$ and $L \;=\; \mathbb{C}^n \times \mathbb{C}$ with the hermitian metric $|(z,\zeta)| = |\zeta| e^{-|z|^2/2}$. One can then naturally identify $\mathcal{H}_k$ with the space

$$\mathcal{H}_k \;=\; \{\, f(z)\, e^{-k|z|^2/2} : \mathbb{C}^n \to \mathbb{C}\,;\; f \text{ is holomorphic and } \int |f(z)|^2\, e^{-k|z|^2}\, dl < \infty \,\}$$

(Bargmann's space), where dl is Lebesgue measure. Here $k = 1/\hbar$. Notice that in the hermitian metric of $L^{\otimes k}$ the constant function equal to one has length $e^{-k|z|^2/2}$. The operators a_j and their adjoints are given by

$$a_j \;=\; \frac{1}{k}\,\frac{\partial}{\partial z_j} \quad \text{and} \quad a_j^* \;=\; z_j\,.$$

All this implies:

Lemma B.1 *The square of the length of the eigenstate $|\nu\rangle$ equals*

$$|\langle z|\nu\rangle|^2 \;=\; \left(\frac{k}{\pi}\right)^n \frac{k^{|\nu|}}{\nu!}\,|z_1|^{2\nu_1}\cdots|z_n|^{2\nu_n}\,e^{-k|z|^2}\,. \tag{134}$$

Let us now plug in (134) in the sum appearing on the left-hand side of Theorem 1, where we replace the index j by multi-indices and

$$E^k_\nu \;=\; \frac{|\nu|+\frac{1}{2}}{k}$$

are the eigenvalues of the harmonic oscillator. We obtain

$$\left(\frac{k}{\pi}\right)^n \sum_\nu \varphi\left(|\nu|+\frac{1}{2}-kE\right)\frac{k^{|\nu|}}{\nu!}\,|z_1|^{2\nu_1}\cdots|z_n|^{2\nu_n}\,e^{-k|z|^2}\,. \tag{135}$$

Let us now specialize to $E=1$ and take φ supported in $(0,1)$ and taking the value one at $1/2$. Then (135) reduces to the sum over eigenfunctions with eigenvalue k, that is

$$\left(\frac{k}{\pi}\right)^n \sum_{\nu\,;\,|\nu|=k}\frac{k^k}{\nu!}\,|z_1|^{2\nu_1}\cdots|z_n|^{2\nu_n}\,e^{-k|z|^2} \;=\; \left(\frac{k}{\pi}\right)^n \frac{k^k}{k!}\,(|z|^2)^k\,e^{-k|z|^2} \tag{136}$$

where we have used the multinomial theorem. Next estimate $k!$ by Stirling's formula, $k!\sim\sqrt{2\pi}k^{1/2}k^ke^{-k}$, to obtain that (136) is asymptotic to

$$\frac{k^{n-1/2}}{\pi^n\sqrt{2\pi}}\,(|z|^2)^k\,e^{-k(|z|^2-1)}\,. \tag{137}$$

A simple analysis shows that $(|z|^2)^k\,e^{-k(|z|^2-1)}$ is rapidly decreasing as $k\to\infty$ unless $|z|=1$, in which case it equals one. Let's summarize the conclusion:

Lemma B.2

$$\sum_{\nu\,;\,|\nu|=k}|\langle z|\nu\rangle|^2 \;\sim\; \begin{cases} O(k^{-\infty}) & \text{if } |z|^2\neq 1\\ \frac{k^{n-1/2}}{\pi^n\sqrt{2\pi}} & \text{if } |z|^2=1\end{cases}$$

We finish by observing that Theorems 1.1 and 2.7 have straightforward proofs in case of the harmonic oscillator. For brevity we will only consider the case $p_1=p_2$. Observe that, in general,

$$\sum_{i=0}^{\infty}\varphi(k(E^k_i-E))\,|\psi^k_i(z)|^2 \;=\; \frac{1}{2\pi}\int\hat{\varphi}(t)\,e^{-iktE}\,\langle z|e^{itkS}|z\rangle\,dt\,. \tag{138}$$

Here $\hat{\varphi}$ is the Fourier transform of φ and

$$\langle z|w\rangle \;=\; \left(\frac{k}{\pi}\right)^n e^{kz\overline{w}}\,e^{-k|z|^2/2}\,e^{-k|w|^2/2}\,. \tag{139}$$

is the reproducing kernel. It is well-known however that for the harmonic oscillator $e^{itkS}|z\rangle \;=\; e^{it/2}\,|e^{-it}z\rangle$, and therefore (138) becomes exactly

$$\left(\frac{k}{\pi}\right)^n \frac{1}{2\pi}\int\hat{\varphi}(t)\,e^{it/2}\,e^{-iktE}\,e^{k|z|^2(\exp(it)-1)}\,dt\,. \tag{140}$$

We can now apply the method of stationary phase. Assuming $E \neq 0$, the critical points are of the form $t = 2\pi\ell$ with $\ell \in \mathbb{Z}$, provided that $|z|^2 = E$ (otherwise there are no critical points). The contribution from such a critical point to the asymptotics of (138) is exactly

$$\frac{k^{n-\frac{1}{2}}}{\pi^n\sqrt{2\pi}}\,\frac{e^{i\pi\ell}}{\sqrt{E}}\,e^{-i2\pi klE}\,\hat{\varphi}(2\pi\ell)\,. \tag{141}$$

In order to compare this result with that of Theorem 2.7, recall that $\mathcal{M}_{2\pi\ell} = (-1)^l I$, and that

$$e^{t\Xi}\,(|0\rangle) \;=\; |-t\Xi\rangle\,.$$

(See [14], formula (1.72) keeping in mind that different normalizations are being used. To check our normalizations, observe that $e^{t\Xi}$ must be unitary and that $\langle z|z\rangle$ is independent of z). The state e is in this case the normalized ground state of the harmonic oscillator, i.e.

$$e \;=\; \pi^{n/2}\,|z=0\rangle\,.$$

Together with (139) (with $k=1$), this implies that

$$\int_{-\infty}^{+\infty}\langle e|\exp t\Xi|e\rangle\,dt \;=\; \int_{-\infty}^{+\infty} e^{-t^2|\Xi|^2/2}\,dt \;=\; \frac{\sqrt{2\pi}}{|\Xi|} \;=\; \frac{\sqrt{2\pi}}{|z|}.$$

(Since the flow of Ξ equals "complex multiplication by e^{it}" we must have that $|\Xi| = |z|$.) If we are at a critical point as above we get

$$\langle\mathcal{M}_{-2\pi\ell}e|P_\Xi e\rangle \;=\; \sqrt{2\pi}\,\frac{e^{i\pi\ell}}{\sqrt{E}}\,.$$

Finally one can check that the angle $\theta_{2\pi\ell}$ is equal to $-2\pi lE$, so (138) agrees with Theorem 2.7 (both the subprincipal symbol of Q and the function c are zero in this case).

C References

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