ON THE PSEUDOSPECTRA OF BEREZIN-TOEPLITZ OPERATORS

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Contents

1. Introduction

The study of pseudospectra of operators is a very active area of research, of importance in applied mathematics and numerical analysis. We recall that if $\epsilon > 0$, the ϵ-pseudospectrum of an operator Q acting on a Hilbert space is the set of complex numbers, $\lambda \in \mathbb{C}$, such that

$$\|(Q - \lambda I)^{-1}\| \geq \frac{1}{\epsilon},$$

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or, equivalently, the set of λ such that

$$\inf_{\psi\neq 0} \frac{\|(Q-\lambda I)(\psi)\|}{\|\psi\|} \leq \epsilon. \tag{1.1}$$

The importance of this set in applied settings stems from the following two facts:

(1) If ϵ is very small it may be difficult to distinguish the ϵ-pseudospectrum from the spectrum of Q,
(2) If Q is strongly non normal the ϵ-pseudospectrum generally is much bigger than the spectrum, even if ϵ is very small.

In case Q is a differential or, more generally, a pseudodifferential operator with a small parameter, $\hbar$ (such as a Schrödinger operator), it is natural to consider the asymptotic behavior of the ϵ-pseudospectrum of Q where ϵ is related to $\hbar$, e.g. $\epsilon = O(\hbar^\infty)$. Dencker, Sjöstrand and Zworski have recently studied this problem by microlocal techniques, [10]. Earlier results for Schrödinger operators were obtained by E. B. Davies, [9], and P. Redparth, [17].

On the other hand, in applications one often deals with large *matrices*, and then it is natural to estimate the pseudospectrum in terms of the size of the matrix. In a recent paper, [18], Trefethen and Chapman considered this problem for matrices $T^{(N)} = (T^N_{jl})$ where

$$T^{(N)}_{jl} = f_{(l-j)\mathrm{mod}N}(j/N), \quad 1 \leq j,l \leq N. \tag{1.2}$$

Here the f_j are 1-periodic coefficient functions . The main result of Trefethen and Chapman is that, under certain assumptions including the following "twist" condition,

$$\Im\Big(\frac{\partial f}{\partial p}/\frac{\partial f}{\partial x}\Big)(x_0,p_0) < 0 \tag{1.3}$$

where $f(x,p) = \sum_j f_j(p)e^{ijx}$, then $\lambda = f(x_0,p_0)$ is in the ϵ-pseudospectrum of T_N where $\epsilon = O(e^{-cN})$ for some $c > 0$.

The purpose of this note is to show that microlocal techniques can also be applied to the study of the pseudospectra of matrices such as (1.2) (and generalizations). In this light we interpret the twist condition (1.3) as Hörmander's solvability condition

$$\{\Re f\,,\,\Im f\}(x_0,p_0) < 0 \tag{1.4}$$

on the Poisson bracket of the real and imaginary parts of the symbol of a pseudodifferential operator. Indeed if we define the Poisson bracket of the variables x and p above to be one, it is easy to check that (1.3) is precisely Hörmander's condition. The connection between Hörmander's condition and pseudospectra was first made by M. Zworski in [21].

In this paper we construct pseudomodes for Berezin-Toeplitz operators under condition (1.4) on the (smooth) symbol. Our construction is symbolic: in §2 we introduce spaces of Hermite distributions containing the pseudomodes. From the point of view of the symbolic calculus of these distributions, condition (1.4) is exactly the condition on ϵ for the operator: $\frac{d}{dx} + \epsilon x$ to have a kernel in the Schwartz space of $\mathbb{R}$, namely $\epsilon > 0$.

Although we will discuss our results in detail in the next section, we should mention some limitations of our work. The methods of Trefethen and Chapman apply to rough symbols, f, and they obtain exponentially small error terms. For analytic symbols, it is very likely that exponentially small estimates (in the Toeplitz setting) can be achieved by microlocal methods, as has been done in [10] for pseudo-differential operators. The problem of dealing with general non-smooth symbols, however, is much more challenging. Trefethen and Chapman's main theorem includes a *global* condition on the symbol (in addition to 1.2), and they present compelling numerical evidence that global conditions on non-smooth symbols are necessary for the existence of "good" pseudomodes (see §8 of [18]). This is a very interesting issue that we do not address here. On the other hand, our results for smooth symbols are fairly general and include a number of cases not covered by the results in [18] (e. g. the "Scottish flag" matrix). Furthermore, the pseudomodes we construct are localized in phase space, sharpening the localization results of [18].

We also mention that more straightforward microlocal methods can be applied to the study of non-periodic versions of (1.2), along the lines of the example in §4.1. More generally, the Berezin-Toeplitz operator calculus opens up the entire spectrum of phase-space methods to study other problems associated with certain sequences of large matrices, and we hope to return to some of these problems in the future.

1.1. **The main results.** The general setting for B-T operators is a Kähler manifold, X, together with a holomorphic hermitian line bundle, $L \to X$ whose curvature is the symplectic form on X. In our general theorems X will be compact. If $f : X \to \mathbb{C}$ is a "classical Hamiltonian", (a smooth function) consider the sequence of operators $T_f = \{T^{(N)}\,, N = 1, 2, \ldots\}$, acting on the space $\mathcal{H}_N$ of holomorphic sections of the tensor power $L^{\otimes N}$, defined by:

$$\mathcal{H}_N \ni \psi \mapsto \Pi_N(f\psi),$$

where $\Pi_N : L^2(X, L^{\otimes N}) \to \mathcal{H}_N$ is orthogonal projection. The sequence, T_f, is the primary example of a Berezin-Toeplitz operator. More generally, one can allow f to depend on N as well, provided the N-dependence admits an asymptotic expansion as $N \to \infty$

$$f(x, N) \sim \sum_{j=0}^{\infty} N^{-j}\, f_j(x)$$

in the C^∞ topology. The function f_0 is then called the principal symbol of the operator. We include explicit examples of all this in §4. General references for the theory of Kähler quantization and Berezin-Toeplitz operators are [3], [4], [8], [20].

For each N $\mathcal{H}_N$ is finite-dimensional, and for N large, by the Riemann-Roch theorem, $\dim \mathcal{H}_N$ is a polynomial in N of degree one-half the dimension of X, $n := \frac{1}{2}\dim X$.

The simplest cases are when X is a either the torus or the complex projective line, for which $n = 1$. Thus the parameter, N, is essentially the dimension of $\mathcal{H}_N$. Moreover, for such X the spaces $\mathcal{H}_N$ have a natural multiplicity-free representation of the circle group, whose eigenvectors form a canonical basis of $\mathcal{H}_N$. We will write down explicitly the matrices corresponding to a B-T operator on these spaces in the next section. It turns out, for example, that the matrices (1.2) are the matrices of B-T operators on X equal to the two-torus.

Our main result is:

Theorem 1.1. *Let X be a compact, $2n$-dimensional quantized Kähler manifold, and let $T_f = \{T^{(N)}, N = 1, 2, \dots\}$ be a Berezin-Toeplitz operator with smooth principal symbol $f : X \to \mathbb{C}$.*
0. *For all $\lambda \in \mathbb{C}$,*

$$\inf_{\psi\in\mathcal{H}_N} \frac{\|(T^{(N)} - \lambda I)(\psi)\|}{\|\psi\|} = \inf_{x\in X} |f(x) - \lambda| + O(1/\sqrt{N}).$$

1. *Assume that $\lambda = f(x_0)$ where $x_0 \in X$ is such that*

$$\{\Re f\,,\,\Im f\}(x_0) < 0. \tag{1.5}$$

Then there exists a sequence of vectors $\{\psi_N \in \mathcal{H}_N\}$ with microsupport precisely $\{x_0\}$ and such that

$$\frac{\|(T^{(N)} - \lambda I)(\psi_N)\|}{\|\psi_N\|} = O(N^{-\infty}). \tag{1.6}$$

2. *On the other hand, if $\lambda = f(x_0)$ and*

$$\{\Re f\,,\,\Im f\}(x_0) > 0, \tag{1.7}$$

then any sequence $\{\psi_N \in \mathcal{H}_N\}$ such that (1.6) holds has microsupport away from $\{x_0\}$.

We say a few words about the definition of microsupport in §2.1.

In the case of more than one degree of freedom (i. e. if the dimension of X is greater than two), there are multiple pseudomodes under condition (1.5). The level set $f^{-1}(\lambda)$ is a symplectic manifold (at least near x_0) and we construct pseudomodes associated with any germ of isotropic submanifold of it containing x_0.

We will also prove an additional result, analogous to Theorem 4 in [10], whose hypotheses hold typically in case λ is on the boundary of the image of the principal symbol. Let $\Re(f) = f_1$, $\Im(f) = f_2$, and for $I \subset \{1, 2\}^m$ denote by f_I the repeated Poisson bracket:

$$f_I = \Xi_{f_{i_1}}\, \Xi_{f_{i_2}}\, \dots\, \Xi_{f_{i_{m-1}}} f_{i_m}$$

where Ξ_g denotes the Hamilton vector field of g. Denoting the order of the Poisson bracket by $|I| = m$, we define the order of a point $x \in X$ as

$$k(x) := \max\{j \in \mathbb{Z}\,;\ f_I(x) = 0 \text{ for all } |I| \le j\,\}. \tag{1.8}$$

Theorem 1.2. *With the same assumptions as the previous Theorem, let* $\lambda \in \partial Image(f)$ *be such that:*

(1) $df_x \neq 0$ *for every* $x \in f^{-1}(\lambda)$.

(2) *The maximum order* $k := \max_{x\in f^{-1}(\lambda)} k(x)$ *is finite.*

Then there exist $C,\ C_1 > 0$ *such that*

$$C_1 N^{-1/2} \geq \inf_{\psi\in\mathcal{H}_N} \frac{\|(T^{(N)} - \lambda I)(\psi)\|}{\|\psi\|} \geq C N^{-\frac{k}{k+1}}. \tag{1.9}$$

The first inequality in (1.9) follows from Part 0 of Theorem 1.1; the second one follows from subelliptic estimates. In the course of the proof of this Theorem we will also show that, in general, the microsupport of a sequence of vectors $\psi_N \in \mathcal{H}_N$ minimizing the Rayleigh quotient in (1.9) (i.e. the optimal pseudomodes) is contained in $f^{-1}(\lambda)$, see Proposition 3.3.

The existence part of Theorem 1.1 is proved by constructing pseudomodes out of a class of distributions that will be defined in the next section. The proof of Theorems 1.1 and 1.2 appear in §3, and §4 is devoted to examples. We present some additional results, including a description of the limit of the numerical range, in §5.

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2. Preliminaries

2.1. **Setup and strategy.** Let $L \to X$ be as in the previous section, and let $P \subset L^*$ denote the unit circle bundle in the dual of the line bundle L. We denote by

$$L^2(P) = \bigoplus_{k\in\mathbb{Z}} L^2_k(P) \tag{2.1}$$

the Fourier decomposition of functions on P under the action of the circle; explicitly $f \in L^2(P)$ is in $L^2_k(P)$ iff $f(e^{is}\cdot p) = e^{iks}f(p)$. We will also need the spaces

$$C^\infty_k(P) = L^2_k(P) \cap C^\infty(P). \tag{2.2}$$

It is a tautology that $C^\infty_k(P)$ (resp. $L^2_k(P)$) can be naturally identified with the space of sections $C^\infty(X, L^{\otimes k})$ (resp. $L^2(X, L^{\otimes k})$. We will henceforth identify these spaces without further comment.

P is the boundary of a strictly pseudoconvex domain (the disk bundle of L^*), and under the natural action of the circle group its Hardy space, $\mathcal{H}$, splits into Fourier components that are naturally isomorphic to the spaces of holomorphic sections $\mathcal{H}_N$:

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N, \qquad \mathcal{H}_N = H^0(X, L^{\otimes N}). \tag{2.3}$$

We denote by $\Pi : L^2(P) \to \mathcal{H}$ the Szegö projector of P, and by $\Pi_N : L^2(P) \to \mathcal{H}_N$ the orthogonal projection onto the summand $\mathcal{H}_N$.

The precise structure of the singularities of Π has been known for some time, thanks to work of Boutet de Monvel and Sjöstrand. We now recall the microlocal structure of Π, as described in [5]. Let α denote the connection form on P, and let $\mathcal{Z} \subset T^*P$ be the manifold

$$\mathcal{Z} = \{\, (p, r\alpha_p)\, ;\, p \in P \;,\; r > 0\,\}.$$

This is a conic symplectic submanifold of T^*P, and Π is a Fourier integral operator of Hermite type associated with the canonical relation

$$\mathcal{Z}^\Delta := \{\, (\zeta, \zeta)\, ;\, \zeta \in \mathcal{Z}\,\}.$$

We will say a few words below about the symbol of Π, referring to [5] for the general theory of Fourier integral operators of Hermite type. (See [20] for a description of Π as a Fourier integral operator with complex phase.)

The overall strategy of our proofs is the observation that much of the asymptotic behavior of a sequence $\{\psi_N \in C^\infty(X, L^{\otimes N})\}$ is encoded by the singularities of the distribution on P, $\psi = \sum_{N=1}^\infty \psi_N \in C^{-\infty}(P)$. For example, we have the following elementary result:

Lemma 2.1. *Given a sequence of vectors $\psi_N \in C^\infty(X, L^{\otimes N})$, let $\psi = \sum_{N=1}^\infty \psi_N \in C^{-\infty}(P)$. For each $s \in \mathbb{R}$, let $H_{(s)}(P)$ denote the Sobolev space on P consisting of distributions, u, such that $(\Delta_P + I)^{s/2}(u) \in L^2(P)$, with the norm $\|u\|_{(s)} = \|(\Delta_P + I)^{s/2}(u)\|_{L^2}$. Then the following are equivalent:*

(a) $\psi \in C^\infty(P)$
(b) *For all $s \in \mathbb{R}$ $\|\psi_N\|_{(s)} = O(N^{-\infty})$.*

If, in addition, $\psi_N \in \mathcal{H}_N$ for all N, then the above are equivalent to:

(c) $\|\psi_N\|_{L^2} = O(N^{-\infty})$.

Proof. The spaces $C_k^\infty(P)$ are invariant under Δ_P and orthogonal in $H_{(s)}$, so for each s $\|\psi\|_{(s)}^2 = \sum_N \|\psi_N\|_{(s)}^2$. It follows that (b) implies that $\psi \in H_{(s)}(P)$ for all s, and therefore it implies (a) (and (c), of course). Assuming (a) now, consider $D_\theta^k \psi$ where k is a positive integer. Since this function is smooth it is in $H_{(s)}(P)$ for each s, and therefore $\|D_\theta^k \psi\|_{(s)}^2 = \sum_N N^{2k} \|\psi_N\|_{(s)}^2 < \infty$, which implies that $\|\psi_N\|_{(s)} = O(N^{-k})$. Therefore (a) implies (b).

Let us define an operator Δ_h by the identity

$$\Delta_P = \Delta_h + D_\theta^2. \tag{2.4}$$

Then $[\Delta_h, D_\theta] = 0$ and the restriction of Δ_h to $C_N^\infty(P)$ agrees exactly with the Laplacian on $C^\infty(X, L^{\otimes N})$ associated with the connection and the Hermitian structure on $L^{\otimes N}$. This has the following consequence: If $\psi_N \in \mathcal{H}_N \subset L^2(P)$,

$$\|\psi_N\|_s = (N^2 + N + 1)^{s/2} \|\psi_N\|_0. \tag{2.5}$$

Indeed elements in $\mathcal{H}_N$ are eigenfunctions of Δ_P: We claim that $\Delta_P(\psi_N) = N(N+1)\,\psi_N$. By virtue of (2.4), this statement is equivalent to: $\Delta_h\psi_N = N\psi_N$. This follows from the well-known Bochner-Kodaira relationship the metric and the $\overline{\partial}$ Laplacian on sections of L^N, see for instance [11] Proposition 6.1. Clearly (2.5) has the consequence that (c) implies (b) for sequences of vectors in $\mathcal{H}_N$. $\square$

We end this subsection with a reminder of the notion of microsupport in Kähler quantization. Let $\{\psi_N \in \mathcal{H}_N\}$ be a sequence of holomorphic sections of the tensor powers of L, and let $\psi = \sum_N \psi_N$. Since $\Pi(\psi) = \psi$, the wave-front set of ψ is included in $\mathcal{Z}$. The microsupport of the sequence is defined as the subset of X which is the projection of $\mathrm{WF}(\psi)$: we say that $x \in X$ is in the microsupport of $\{\psi_N\}$ if and only if

$$\exists p_x \in P \text{ such that } \pi(p_x) = x \text{ and } (p, \alpha_p) \in \mathrm{WF}(\psi).$$

It follows from the above that microsupport of the sequence is the empty set iff $\|\psi_N\|_{L^2} = O(N^{-\infty})$. In addition, one can show that $x_0 \in X$ is not in the microsupport of $\{\psi_N \in \mathcal{H}_N\}$ iff there exists a neighborhood, V, of x_0 such that $\sup_{x\in V} |\psi_N(p_x)| = O(N^{-\infty})$, where for all x $p_x \in P$ denotes any point projecting to x.

The microsupport has a characterization in terms of the action of Toeplitz operators analogous to the characterization of the ordinary wave-front set by the action of pseudodifferential operators. We refer to [8], §5, for alternative descriptions of the microsupport.

2.2. **Polarized Hermite distributions.** In this section we define and analyze the concept of generalized wave packets and their symbols in the context of Kähler quantization. In fact we'll define more general states, associated to isotropic submanifolds of a quantized Kähler manifold (although everything we do generalizes to any almost Kähler manifold quantized by a projector, Π, with the same microlocal structure as the Szegö projector.)

Definition of polarized Hermite distributions.

Let us begin by considering a closed conic isotropic submanifold

$$\mathcal{R} \subset \mathcal{Z}.$$

Obviously $\mathcal{R}$ is isotropic in $T^*P \setminus \{0\}$, and therefore associated with it are spaces $I^l(P, \mathcal{R})$ of Hermite distributions on P. The general theory of such distributions (together with many applications) was developed by Boutet de Monvel and Guillemin, see (see [5], or the Appendix for additional remarks). The polarized Hermite distributions associated with $\mathcal{R}$ are simply the projections of elements of $I^l(P, \mathcal{R})$ by the Szegö projector:

Definition 2.2. The space of polarized Hermite distributions of order l associated with $\mathcal{R}$ is

$$I^l_\Pi(P, \mathcal{R}) := \Pi(I^l(P, \mathcal{R})). \tag{2.6}$$

We should point out that by the composition Theorem 9.4 of [5] one has the inclusion: $I^l_{\Pi}(P,\mathcal{R}) \subset I^l(P,\mathcal{R})$.

Notice that one has a natural isomorphism

$$\begin{array}{ccc} P\times\mathbb{R}^+ & \to & \mathcal{Z} \\ (p,r) & \mapsto & (p, r\alpha_p) \end{array}$$

which becomes a symplectomorphism if we put on $P\times\mathbb{R}^+$ the symplectic structure

$$-d(r\alpha) = -r\,d\alpha - \alpha\wedge dr \tag{2.7}$$

(here r is the coordinate on the $\mathbb{R}^+$ factor). Using this description of $\mathcal{Z}$ it is easy to show that the base of the cone, $\mathcal{R}$, is a submanifold $\tilde{Y}\subset P$ such that the infinitesimal generator of the circle action on P is never tangent to $\tilde{Y}$. Moreover, the projection, $\pi : P\to X$, restricts to an isotropic immersion of $\tilde{Y}$ in X. Let us denote by Y this immersed isotropic submanifold of X.

Definition 2.3. A sequence of holomorphic sections, $\{\psi_N\in\mathcal{H}_N\}$, of the tensor powers of L will be called an *Hermite state associated with* Y iff there exists $\psi\in I^l_{\Pi}(P,\mathcal{R})$ such that the Fourier components of ψ according to (2.3) are precisely the ψ_N.

For example, if Y is a single point then the coherent states at that point are an Hermite state. The case when Y is Lagrangian (and hence $\tilde{Y}$ Legendrian) was considered in [7].

Symbolic matters.

Our next step is to define the symbol of a polarized Hermite distribution. We begin by recalling the nature of the symbol of general elements in $I^l(P,\mathcal{R})$. Such distributions have symbols which are *symplectic spinors* associated with $\mathcal{R}$. The definitions are made in the tangent space to T^*P, so for notational convenience for each $\rho\in\mathcal{R}$ we'll set

$$R_\rho = T_\rho\mathcal{R}, \qquad Z_\rho = T_\rho\mathcal{Z}.$$

Let

$$N_\rho := R_\rho^\circ / R_\rho$$

where R_ρ° denotes the symplectic orthogonal of R_ρ inside $T_\rho(T^*P)$. N_ρ is a symplectic vector space, called the symplectic normal space to $\mathcal{R}$ at ρ. Abstractly, the symbol of $u\in I^l(P,\mathcal{R})$ at ρ is a smooth vector in the metaplectic representation of the metaplectic group of N_ρ tensored with a half-density along R_ρ, i.e. an element of:

$$\mathrm{Spin}(R_\rho) := \textstyle\bigwedge^{1/2}(R_\rho)\otimes H_\infty(N_\rho).$$

Lemma 2.4. *Let $E_\rho = \{v\in Z_\rho\ ;\ \forall u\in R_\rho\ \omega(u,v)=0\}/R_\rho$ be the symplectic normal of $\mathcal{R}$ in $\mathcal{Z}$ at ρ. Then, the symplectic normal, N_ρ, is naturally isomorphic to the direct sum*

$$N_\rho = E_\rho\oplus Z_\rho^\circ \tag{2.8}$$

where Z_ρ° *is the symplectic orthogonal of* Z_ρ *in* $T_\rho(T^*P)$.

Proof. This follows from the fact that

$$R_\rho^\circ = R_\rho^{\circ \mathcal{Z}} \oplus Z_\rho^\circ$$

where $R_\rho^{\circ \mathcal{Z}}$ is the symplectic orthogonal of R_ρ inside Z_ρ, which itself follows from the fact that $\mathcal{Z}$ is a symplectic submanifold of T^*P. □

It is important to note that the projection $T_\rho(T^*P) \to T_{\pi(\rho)}X$ induces a symplectic isomorphism $Z_\rho^\circ \simeq T_{\pi(\rho)}X$. On the other hand, because of the negative sign in (2.7), the same projection takes E_ρ to $(T_{\pi(\rho)}Y^\circ / T_{\pi(\rho)}Y)^-$, where the minus indicates a reversal of the symplectic structure.

It follows from Lemma 2.4 that the metaplectic representation of the metaplectic group of N_ρ is a tensor product:

$$H(N_\rho) = H(E_\rho) \hat{\otimes} H(Z_\rho^\circ)$$

(Hilbert space tensor product). It is this decomposition that reveals the structure of the symbol of a polarized Hermite distribution. In order to discuss this structure, we recall that the smooth-vector factor of the symbol of Π is of the form $e \otimes \overline{e}$, where $e \in H_\infty(Z_\rho^\circ)$ is a normalized "ground state" (which can be identified with the ground state of the harmonic oscillator on $T_{\pi(\rho)}X$ defined by the metric).

Proposition 2.5. *The symbol of a polarized Hermite distribution,* $u \in I_\Pi(P, \mathcal{R})$, *is of the form:*

$$\sigma_u = \nu_u \otimes \kappa_u \otimes e,$$

where $e \in H_\infty(Z_\rho^\circ)$ *is the symbol of the polarization and*

$$\nu_u \in \bigwedge\nolimits^{1/2}(R_\rho) \quad \kappa_u \in H_\infty(E_\rho).$$

By dividing σ_u *by* e *one obtains the non-trivial map in the following exact sequence:*

$$0 \to I_\Pi^{l-1/2}(P, \mathcal{R}) \hookrightarrow I_\Pi^l(P, \mathcal{R}) \to \bigwedge\nolimits^{1/2}(R_\rho) \otimes H_\infty(E_\rho) \to 0.$$

We relegate the proof of this technical proposition to an appendix.

3. Proofs

3.1. **Proof of Theorem 1.1.** Let $T_f = \{T^{(N)}, N = 1, 2, \ldots\}$ be a Berezin-Toeplitz operator with smooth principal symbol $f : X \to \mathbb{C}$.

Part (0) of Theorem 1.1 is not difficult. Without loss of generality we can assume that $\lambda = 0$. Since $\{(T^{(N)})^* T^{(N)}\}$ is a Toeplitz operator with symbol $|f|^2$, one has that for all $\psi_N \in \mathcal{H}_N$,

$$\|T^{(N)}\psi_N\|^2 = \langle (T^{(N)})^* T^{(N)} \psi_N, \psi_N \rangle \sim \langle \Pi_N(|f|^2 \psi_N), \psi_N \rangle = \langle |f|^2 \psi_N, \psi_N \rangle.$$

More precisely, from the definition of B-T operators we have that for any sequence of $\psi_N \in \mathcal{H}_N$,

$$\|T^{(N)}\psi_N\|^2 = \int_X |f(x)|^2\, |\psi_N(x)|^2\, d\mu_x + \|\psi_N\|^2 \cdot O(1/N),$$

where $d\mu$ is the measure on X and $|\psi_N(x)|^2$ is the square of the length of $\psi_N(x)$ in the Hermitian norm of $L^{\otimes N} \to X$. It follows that

$$\inf_{\psi_N \in \mathcal{H}_N \setminus \{0\}} \frac{\|(T^{(N)} - \lambda I)(\psi_N)\|}{\|\psi_N\|} \geq \inf_{x\in X} |f(x)| + O(1/\sqrt{N}).$$

In the other direction, let x_0 be the point of X where $\inf|f(x)|$ is attained, and let $\varphi^{(N)}_{p_0} = \Pi(\cdot, p_0)$ be a coherent state at a point $p_0 \in P$ that projects down to x_0. Then

$$\inf_{\psi_N \in \mathcal{H}_N \setminus \{0\}} \frac{\|T^{(N)}\psi_N\|^2}{\|\psi_N\|^2} \leq \frac{\langle (T^{(N)})^* T^{(N)} \varphi_{p_0}, \varphi_{p_0}\rangle}{\|\varphi_{p_0}\|^2} = |f(p_0)|^2 + O(1/N).$$

We now prove part (1) of the Theorem. We begin with a few preliminary considerations. By Proposition 2.13 of [5], there exists a classical pseudodifferential operator of order zero on the circle bundle P, Q, such that:

(1) Q commutes with the Szegö projector ($[\Pi\,,\, Q] = 0$) and with the S^1 action.
(2) For each N the restriction of Q to $\mathcal{H}_N$, $Q : \mathcal{H}_N \to \mathcal{H}_N$ is equal to $T^{(N)}$.
(3) The principal symbol of Q satisfies:

$$\forall (p, r\alpha_p) \in \Sigma \qquad \sigma_Q(p, r\alpha_p) = f(\pi(p))$$

where $\pi : P \to X$ is the projection.

Assume now that $\lambda = f(x_0)$, where $x_0 \in X$ is such that

$$\{\Re f\,,\, \Im f\}(x_0) < 0. \tag{3.1}$$

The inverse image $f^{-1}(\lambda)$ is, in a neighborhood of x_0, a codimension-two symplectic submanifold of X. Let us pick an isotropic submanifold of $f^{-1}(\lambda)$ (not necessarily closed), Y, containing x_0. Then Y is an isotropic submanifold of X. Our considerations are local: we restrict our attention to a neighborhood of x_0 where $\{\Re f\,,\, \Im f\}$ is negative and such that there exists a lift of Y to a conic isotropic submanifold, $\mathcal{R} \subset \mathcal{Z}$, in the sense of the previous section. We will construct a pseudomode with microsupport equal to Y. Notice that we may take $Y = \{x_0\}$ if we wish.

Since $[\Pi, Q] = 0$ and Π is self-adjoint, we have $[\Pi, Q^\dagger] = 0$. Therefore, by Proposition 11.4 of [5],

$$\forall (p, r\alpha_p) \in \Sigma \qquad \{\Re\sigma_Q\,,\, \Im\sigma_Q\}(p, r\alpha_p) = \{\Re f\,,\, \Im f\}(\pi(p)), \tag{3.2}$$

where σ_Q is the principal symbol of Q. Notice that the Poisson bracket on the left is on T^*P (with respect to the cotangent bundle structure), while the one on the right is the Poisson bracket on X. Therefore, the Poisson bracket conditions (3.1, 3.5) on f are inherited by σ_Q.

To see how the Poisson bracket condition becomes relevant in our considerations, we first consider a calculation in the Heisenberg representation on $L^2(\mathbb{R}^k)$.

Lemma 3.1. *Let $\mathcal{L}$ be the operator on $L^2(\mathbb{R}^k)$ corresponding to the action of a vector $\xi \in (\mathbb{R}^{2k}, \omega) \otimes \mathbb{C}$ under the Heisenberg representation. If*

$$\omega(\Re\xi, \Im\xi) > 0,$$

then the restriction of $\mathcal{L}$ to the smooth vectors maps $\mathcal{S}(\mathbb{R}^k)$ onto itself, with a non-zero kernel.

Proof. The metaplectic representation describes how $\mathcal{L}$ transforms under the action of the symplectic group on ξ. That is, $\mathcal{L}_{g.\xi} = U(g)\mathcal{L}_\xi U(g^{-1})$, where $g \mapsto U(g)$ is the projective unitary representation that gives rise to the metaplectic representation when we take the double cover. The action of the metaplectic group preserves the smooth vectors $\mathcal{S}(\mathbb{R}^k)$, so in our argument we can replace ξ by $g.\xi$ for g symplectic.

Under the assumption that $\epsilon := \omega(\Re\xi, \Im\xi) > 0$, it is a straightforward exercise to see that g can be chosen so that $g.\xi = \epsilon e_1 + i f_1$, where $e_1, \dots, e_k, f_1, \dots, f_k$ is the standard symplectic basis for $\mathbb{R}^{2k}$. Thus it suffices to prove the result for

$$\mathcal{L} = \frac{\partial}{\partial x_1} + \epsilon x_1.$$

We see then that $\ker \mathcal{L} \cap \mathcal{S}(\mathbb{R}^k)$ contains functions of the form $\psi(x_1, \dots, x_k) = e^{-\epsilon x_1^2/2} a(x_2, \dots, x_k)$. And to show that $\mathcal{L}$ maps $\mathcal{S}(\mathbb{R}^k)$ onto itself, let $f \in \mathcal{S}(\mathbb{R}^k)$. The ODE $\mathcal{L}u = f$ can be solved by variation of parameters:

$$u(x_1, \dots, x_k) = \int_0^{x_1} f(t, x_2, \dots, x_k) e^{\epsilon(t^2 - x_1^2)/2}\, dt.$$

For $\epsilon > 0$, the estimates showing that $u \in \mathcal{S}(\mathbb{R}^k)$ follow easily. □

Applied at the symbol level, Lemma 3.1 leads directly to the following construction:

Proposition 3.2. *There exists a distribution u in the class $I^0_\Pi(P, \mathcal{R})$ of polarized Hermite distributions of order zero associated with $\mathcal{R}$ such that:*

$$(Q - \lambda I)u \in C^\infty(P). \tag{3.3}$$

Proof. Suppose $u \in I^0_\Pi(P, \mathcal{R})$ with $\sigma_u = \nu_u \otimes \kappa_u \otimes e$ as in Proposition 2.5. It's clear that $(Q - \lambda I)u = \Pi(Q - \lambda I)u$ is also a polarized Hermite distribution. What is its symbol? Since the principal symbol of $Q - \lambda I$ vanishes on $\mathcal{R}$, we are led to use the first transport equation for the Hermite calculus. The Hamilton vector field, ξ, of the symbol of Q at a point $\rho \in \mathcal{R}$ is in the symplectic normal space N_ρ. Therefore, the Heisenberg representation of that space associates to ξ_ρ an operator, $\mathcal{L}$, on the space $H_\infty(N_\rho)$. According to Theorem 10.2 in [5], $(Q - \lambda I)(u) \in I^{-1/2}(P, \mathcal{R})$ and its symbol is $\nu_u \otimes \mathcal{L}(\kappa_u \otimes e)$. In fact, since $(Q - \lambda I)u$ is still polarized, under the decomposition $H_\infty(N_\rho) \simeq H_\infty(E_\rho) \otimes H_\infty(Z^\circ)$, $\mathcal{L}$ acts only on $H_\infty(E_\rho)$. The symbol is really $\nu_u \otimes \mathcal{L}(\kappa_u) \otimes e$.

We noted in the previous section that the pull-back of the symplectic form from X under the natural projection $E_\rho \to T_{\pi(\rho)}X$ is the opposite of the symplectic form on E. So the Poisson bracket condition (3.2) along with Lemma 3.1 implies that, as an operator on $H_\infty(E_\rho)$, $\mathcal{L}$ is onto and has a non-trivial kernel.

Because $\mathcal{L}$ has a kernel, we can choose $u_0 \in I^0(P,\mathcal{R})$ with symbol $\mathcal{L}(\sigma(u_0)) = 0$. Therefore $v_1 := (Q-\lambda I)(u_1) \in I^{-1}(P,\mathcal{R})$. Because $\mathcal{L}$ maps onto $H_\infty(E_\rho)$ (by Lemma 3.1 again), we can then find $u_1 \in I^{-1/2}(P,\mathcal{R})$ such that $\mathcal{L}(\sigma_{u_1}) = -\sigma_{v_1}$, Thus $v_2 := (Q-\lambda I)(u_0+u_1) \in I^{-3/2}(P,\mathcal{R})$. Continuing in this fashion and finishing with a Borel summation of the u_j's produces $u \in I^0(P,\mathcal{R})$ such that $(Q-\lambda I)(u)$ is of order $(-\infty)$, and therefore smooth. □

We can now finish the proof of the part (1) of Theorem 1.1. Let u be as in the previous Proposition, which we choose to have a Gaussian principal symbol. Define $u_N := \Pi_N(u)$. By Lemma 2.1, (3.3) implies the norm estimates

$$\|(T^{(N)} - \lambda I)u_N\| = O(N^{-\infty}). \tag{3.4}$$

On the other hand, letting $\varphi_{p_0}^{(N)}$ be the coherent state at $p_0 \in P$, from the reproducing property $u_N(p_0) = \langle \varphi_{p_0}^{(N)}, u_N\rangle$ we obtain the estimate

$$\|u_N\| \geq \frac{|u_N(p_0)|}{\|\varphi_{p_0}^{(N)}\|}.$$

Combining (A.1) with the asymptotics

$$\|\varphi_{p_0}^{(N)}\| = \Pi_N(p_0,p_0) \sim \left(\frac{N}{2\pi}\right)^n$$

(see e. g. equation (28) in [20]), we see that

$$\|u_N\| \geq CN^{-(l+1)/2},$$

where $l = \dim Y$. Together with (3.4) this implies (1.6).

To prove part (2) of Theorem 1.1, we start by assuming that

$$\{\Re f\,,\, \Im f\}(x_0) > 0, \tag{3.5}$$

and let $u_N \in \mathcal{H}_N$ be a sequence of vectors such that

$$\|(T^{(N)} - \lambda I)(u_N)\| = O(N^{-\infty}) \quad \text{and} \quad \|u_N\| = 1.$$

If we let u be the distribution on P whose Fourier coefficients are the u_N, then we can rewrite this condition as

$$(Q - \lambda I)(u) \in C^\infty(P). \tag{3.6}$$

We will now quote Theorem 27.1.11. of [13] (Hörmander, Vol IV) asserting that the Poisson bracket condition (3.5), translated to the corresponding statement about σ_Q, implies that $Q - \lambda I$ is microlocally subelliptic on the set

$$\Sigma_{x_0} := \{\rho \in \mathcal{Z};\ \pi(\rho) = x_0\} \subset T^*P,$$

with loss of 1/2 derivatives. Therefore, by (3.6), for all $s \in \mathbb{R}$, $u \in H^{\mathrm{loc}}_{(s)}$ at every $\rho \in \Sigma_{x_0}$. What this means is that for each such ρ we can write: $u = u_0 + u_1$ where $u_1 \in H_{(s)}(P)$ and $\rho \notin \mathrm{WF}(u_0)$, for all s. Therefore the wave-front set of u is disjoint from Σ_{x_0}.

3.2. **Proof of Theorem 1.2.** There are two ingredients in the proof, one is a very general localization statement (well-known in the theory of $\hbar$-admissible ΨDOs), and the second Hörmander's results on microlocal subellipticity that we used in the previous section. We begin with the localization result, which is of interest in its own right.

Proposition 3.3. *Let $T_f = \{T^{(N)}\}$ be a Berezin-Toeplitz operator with principal symbol f, and let $\{u_N \in \mathcal{H}_N\}$ be a minimizing sequence of the Rayleigh quotients, $\{\frac{\|(T^{(N)}-\lambda I)(\psi)\|}{\|\psi\|}, \psi \in \mathcal{H}_N\}$, where $\|u_N\| = 1$ for all N. Then the distribution $u := \sum_N u_N \in C^{-\infty}(P)$ has wave-front set contained in the set of points, $\rho \in T^*P$, such that $\pi(\rho) \in f^{-1}(\lambda)$, and therefore the microsupport of $\{u_N\}$ is contained in $f^{-1}(\lambda)$.*

Proof. For simplicity of notation, assume without loss of generality that $\lambda = 0$. Let Q denote the operator on P inducing the $T^{(N)}$ and commuting with Π, as before. The minimizing sequence is a sequence of eigenstates of the non-negative, self-adjoint classical pseudodifferential operator of order zero $S = Q^*Q$ on P. We will denote by $S^{(N)} : \mathcal{H}_N \to \mathcal{H}_N$ the restriction of S to $\mathcal{H}_N$. By assumption u_N is an eigenvector of $S^{(N)}$ corresponding to the smallest eigenvalue. Let $\chi \in C_0^\infty(\mathbb{R})$ be a test function, R a zeroth-order ΨDO on P, and consider the traces

$$\Upsilon^{(N)}_{\chi,R} = \mathrm{Tr} \int \chi(t)\, R\, e^{-itNS_N}\, dt.$$

If we write the eigenvalues and eigenvectors of S_N in the form:

$$S_N(\psi_j^{(N)}) = E_j^{(N)}\, \psi_j^N, \quad E_1^{(N)} \leq E_2^{(N)} \leq \cdots\ E_{d_N}^{(N)}$$

where $d_N = \dim \mathcal{H}_N$ and $\{\psi_j^N\}_j$ is an orthonormal basis of $\mathcal{H}_N$, then

$$\Upsilon^{(N)}_{\chi,R} = \sum_{j=1}^{d_N} \hat{\chi}(NE_j^{(N)})\, \langle R\psi_j^{(N)}, \psi_j^{(N)}\rangle.$$

Just as in the proof of part (0) of Theorem 1.1, by taking coherent states as trial functions one obtains the estimate: $E_1^{(N)} = O(1/N)$.

Lemma 3.4. *If the microsupport of R is disjoint from the characteristic set of Q, then $\Upsilon^{(N)}_{\chi,R} = O(N^{-\infty})$.*

Proof of the Lemma: The operator $e^{-itD_\theta S}$ is a Fourier integral operator, and a simple wave-front set calculation shows that the wave-front set of the operator

$$\Pi \circ S_\chi := \Pi \circ \int \chi(t)\, e^{-itD_\theta S}\, dt$$

is contained in the set

$$\{(\rho,\rho')\in \mathcal{Z}\times\mathcal{Z}\ ;\ \sigma_Q(\rho)=0=\sigma_Q(\rho')\}.$$

Therefore, if the microsupport of R is disjoint from the characteristic set of Q, the operator $R\circ\Pi\circ S_\chi$ is smoothing. Consider next the generating function of the $\Upsilon^{(N)}$,

$$\Upsilon(s):=\sum_N e^{iNs}\,\Upsilon^{(N)}_{\chi,R}=\operatorname{tr}\Big(U(e^{is})\circ R\circ\Pi\circ S_\chi\Big),$$

where $U(e^{is})$ is the operator on P given by composition by the action of e^{-is}. Another wave-front set calculation shows that $\Upsilon\in C^\infty(S^1)$ because $R\circ\Pi\circ S_\chi$ is smoothing, and therefore the Fourier coefficients of Υ are rapidly decreasing. This proves the lemma

Continuing with the proof of the Proposition, choose χ so that $\hat\chi>0$ (we can even take it so that $\hat\chi$ is equal to one in a neighborhood of zero), and choose R of the form: $R=F^*\circ F$ where F is a zeroth order ΨDO on P. Then all terms in the sum defining $\Upsilon^{(N)}_{\chi,R}$ are non-negative, and the previous Lemma implies that, if the microsupport of F is disjoint from $\mathrm{Char}(Q)$, one has: $\|F(u_N)\|=O(N^{-\infty})$. But this implies that, for any such F,

$$F(u)=\sum_N F(u_N)\in C^\infty(P).$$

Since we can pick F microlocally elliptic in a neighborhood of any point in the complement of $\mathrm{Char}(Q)$, the wave-front set of u must be contained in $\mathrm{Char}(Q)$. $\square$

Turning to the proof of Theorem 1.2, we note that under its assumptions we are in a position to apply the subellipticity results of Hörmander. Note first that any given repeated Poisson bracket of the real and imaginary parts of f evaluated at a point $x\in X$ is equal to the same repeated Poisson bracket of the real and imaginary parts of the principal symbol of Q, evaluated at any $\rho\in\mathcal{Z}$ such that $\pi(\rho)=x$.

Next, we claim that the hypotheses of Theorem 1.2 imply the hypotheses of Theorem 27.1.11. of [13], namely:

(A) For every $x\in f^{-1}(\lambda)$ there is some $j\le k$ and some $z\in\mathbb{C}$,

$$\Big((\Xi_{\Re(zf)})^j\,\Im(zf)\Big)(x)\neq 0 \tag{3.7}$$

where $\Xi_{\Re(zf)}$ is the Hamilton vector field of $\Re(zf)$, considered as a differential operator.

(B) The repeated Poisson bracket above is non-negative (it is positive, actually) if j is the smallest integer such that (3.7) holds for some z. Moreover such j is odd.

Part (A) follows from hypothesis (2) in Theorem 1.2 by Corollary 27.2.4 in [13]. Lemma 5.1 of [10] shows that hypothesis (1) of Theorem 1.2 implies Hörmander's Condition $(\overline{\Psi})$, which, as indicated in the remark after Theorem 27.1.11 of [13], implies (B). The proof of this remark is further detailed in the first paragraph in the proof of Theorem 4 of [10].

Let $\Theta \subset \mathcal{Z}$ be the set of points in $\mathcal{Z}$ projecting to $f^{-1}(0)$ (recall that we took $\lambda = 0$). We conclude, by Theorem 27.1.11 of [13] and the previous considerations, that for all $\rho \in \Theta$, Q is subelliptic at ρ with loss of at most $\delta = k/(k+1)$ derivatives.

By Lemma 27.1.5 of [13], for each $\rho \in \Theta$ there exists a zeroth-order ΨDO, A_ρ, non-characteristic at ρ and such that

$$\forall g \in C^\infty(P) \quad \|A_\rho g\|_{(-\delta)} \le C_\rho\Big(\|Qg\|_{L^2} + \|g\|_{(-1)}\Big).$$

An examination of the proof of this lemma shows that we can take the symbol of each A_ρ to be non-negative. Since Θ is a cone with compact base, there exists an integer K such that the sum of K of the operators A_ρ, call them $A_1, \dots, A_K$, is non-characteristic at each $\rho \in \Theta$. We denote such a sum by $A = \sum_{j=1}^K A_j$. Then

$$\forall g \in C^\infty(P) \quad \|Ag\|_{(-\delta)} \le C\Big(\|Qg\|_{L^2} + \|g\|_{(-1)}\Big) \tag{3.8}$$

for some fixed $C > 0$. By averaging with respect to the S^1 action on P (which preserves Sobolev norms), we can further assume without loss of generality that $[A, D_\theta] = 0$.

Let B be a microlocal parametrix of A in a neighborhood of Θ such that $[B, D_\theta] = 0$. Since, by Proposition 3.3, the wave-front set of u is contained in Θ, we have:

$$u = BA(u) + g, \quad \text{with } g \in C^\infty(P).$$

By taking Fourier components it follows that

$$\forall N \quad \|u_N\|_{(-\delta)} \le C_B\|A(u_N)\|_{(-\delta)} + \|g_N\|_{(-\delta)},$$

where C_B is the $H_{(-\delta)}$ norm of B. This together with (3.8) implies that for all N

$$\|Q(u_N)\|_{L^2} \ge C_1\|u_N\|_{(-\delta)} - C_2\|g_N\|_{(-\delta)} - \|u_N\|_{(-1)}.$$

Since $g \in C^\infty(P)$, $\|g_N\|_{(-\delta)} = O(N^{-\infty})$, while equation (2.5) gives us that

$$C_1\|u_N\|_{(-\delta)} - \|u_N\|_{(-1)} = C_1(N^2+N+1)^{-\delta/2} - (N^2+N+1)^{-1/2}$$

Since $\delta = k/(1+k) < 1$, this proves Theorem 1.2.

4. Examples

We now look at specific examples of quantized Kähler manifolds, X, and of Toeplitz operators. The corresponding Hilbert spaces have canonical bases and therefore the Toeplitz operators become sequences of matrices of a specific type that we compute.

4.1. **A preliminary example.** We begin with a concrete example associated with $X = \mathbb{C}$, the plane with its usual complex structure. Although this X is not compact (and therefore it does not fit precisely into the general framework of this paper) we will "cut it" to the unit disk, both symplectically and quantum-mechanically. This leads to a sequence of large matrices to which microlocal methods apply, provided one stays away from the boundary of the unit disk. We only consider an explicit operator which is the microlocal model of the general case. It will be clear that what we do easily generalizes to other operators in this setting. In this section we want to be explicit and avoid using the general machinery.

Recall that the Kähler quantization of the plane gives rise to the Bargmann spaces

$$\mathcal{B}_N = \{\, f : \mathbb{C} \to \mathbb{C} \text{ entire }; \ \|f\|^2 := \frac{1}{\pi}\int_X |f(z)|^2\, e^{-Nz\overline{z}}\, dxdy < \infty\}$$

where $z = x + iy$ and $N > 0$. Elements of $\mathcal{B}_N$ arise from the general Kähler quantization scheme applied to $\mathbb{C}$. The quantizing line bundle $L \to \mathbb{C}$ is holomorphically trivial, and so its sections can be identified with entire functions on $\mathbb{C}$. The Hermitian structure on L, however, is not trivial. We introduce the following notation for the length function of $\psi \in \mathcal{B}_N$ as a section of $L^N \to \mathbb{C}$:

$$|\psi(z)|_s := |\psi(z)|\, e^{-Nz\overline{z}/2}. \tag{4.1}$$

Notice that then the norm of ψ is the integral of the function $|\psi|_s$ with respect to the area form.

A fundamental operator on $\mathcal{B}_N$ is the harmonic oscillator (shifted by 1/2, for convenience),

$$\mathrm{Op}(H_{\mathrm{H.O.}}) = N^{-1} z \frac{d}{dz}$$

which is a Berezin-Toeplitz operator with symbol $H_{\mathrm{H.O.}}(z,\overline{z}) = z\overline{z}$. The eigenfunctions and eigenvalues of $\mathrm{Op}(H_{\mathrm{H.O.}})$ are:

$$N^{-1} z \frac{d}{dz} z^j = N^{-1} j z^j, \qquad \text{and} \quad \|z^j\| = N^{-1(j+1)/2}\sqrt{j!},$$

so that

$$|k\rangle = \frac{N^{(k+1)/2}}{\sqrt{k!}}\, z^k, \qquad k = 0, 1, \ldots$$

is an orthonormal basis of $\mathcal{B}_N$. The unit disk is the region of phase space where the classical energy $z\overline{z}$ is less than one. The analogous object quantum-mechanically is the span of the eigenfunctions with eigenvalue less than one, that is the monomials z^j with $j \leq N$. Thus

$$\mathcal{H}_N = \{\, \text{polynomials in the complex variable } z \text{ of degree } \leq N \,\}$$

where it is now natural to restrict N to be an integer. This setting is close to the case of the sphere to be considered in the next section, provided one does not get too close to the boundary of the unit disk. The Hilbert space $\mathcal{H}_N$ of the sphere can also be identified with the space of polynomials in a complex variable of degree at most N, although on the sphere $\|z_j\|$ is essentially $(C_N^j)^{-1/2}$. (Here C_N^j is a notation for the standard binomial coefficients.) Symplectically, the sphere is the disk with its boundary collapsed to a point.

The example we will be studying is based in the following observation. Suppose one has a Berezin-Toeplitz operator, Q, and a state ψ such that $Q\psi = 0$. Suppose one has a "good" semi-classical cut-off operator, Θ_N (a projector), which is semi-classically the identity in a certain region of the plane (referred to as the allowed region), and let $P_N = \Theta_N Q \Theta_N$. Then

$$0 = \Theta_N Q \Theta_N(\psi) + \Theta_N Q \Theta_N^{\perp}(\psi) + \Theta_N^{\perp} Q \Theta_N(\psi) + \Theta_N^{\perp} Q \Theta_N^{\perp}(\psi)$$

where $\Theta_N^{\perp} = I - \Theta_N$ is the complementary projection. The second and third terms in this sum will be very small due to the assumed localization properties of Θ_N. It follows that $\Theta_N(\psi)$ is a good pseudo-mode for P_N (with pseudo-eigenvalue zero) if

$$\frac{\|\Theta_N^{\perp}\psi\|}{\|\Theta_N\psi\|}$$

is small, that is, if ψ concentrates in the classically allowed region of Θ_N. We will take Θ_N to be the orthogonal projection

$$\Theta_N : \ \mathcal{B}_N \to \mathcal{H}_N,$$

for which the classically allowed region is the interior of the unit disk. We now proceed to make this statement more precise. Our basic tool is the reproducing kernel of $\mathcal{B}_N$, in the form of the coherent states: For each $w \in \mathbb{C}$, let

$$\varphi_w(z) := N e^{Nz\overline{w}} \in \mathcal{B}_N. \tag{4.2}$$

These states have the reproducing property

$$\forall \psi \in \mathcal{B}_N, \ z \in \mathbb{C} \qquad \psi(z) = \langle \psi, \varphi_z \rangle. \tag{4.3}$$

Notice that, in particular

$$\|\varphi_w\|^2 = \langle \varphi_w, \varphi_w \rangle = N\, e^{N|w|^2}. \tag{4.4}$$

Lemma 4.1. *For every $\delta > 0$ there exists $n > 0$ such that for all $N > n$ and for each $w \in \mathbb{C}$ such that $|w| < 1$, we have:*

$$\|\Theta_N^{\perp}\varphi_w\|^2 \le \frac{1+\delta}{\sqrt{2\pi}} \|\varphi_w\|^2\, |w|^2 \sqrt{N}\, e^{-N(1-|w|^2)^2/2}. \tag{4.5}$$

Proof. From the Taylor series expression for φ_w and the orthogonality relations of the monomials z^j, one can show that

$$\|\Theta_N^{\perp}\varphi_w\|^2 = \|\varphi_w\|^2 \left(1 - \frac{\Gamma(N+1, N|w|^2)}{N!}\right), \tag{4.6}$$

where $\Gamma(n,x) = \int_x^{\infty} t^{n-1} e^{-t} dt$ is the incomplete gamma function. To estimate the quantity in parenthesis, notice that

$$N! = \Gamma(N+1) = \Gamma(N+1, N|w|^2) + \int_0^{N|w|^2} e^{-t}\, t^N\, dt.$$

Therefore, dividing by $N!$ and making the change of variables: $s = t/N$, we see that the quantity in parenthesis in (4.6) equals

$$\frac{N^{N+1}}{N!}\int_0^{|w|^2} e^{-Ns}\, s^N\, ds = \frac{N^{N+1}}{N!}\, e^{-N}\int_0^{|w|^2} e^{-Nf(s)}\, ds$$

where $f(s) = s - \log(s) - 1$. This is a decreasing function on $s \in (0,1)$, and therefore the last integrand is maximal at $s = |w|^2$. On the other hand, it is elementary to check that

$$\forall \epsilon \in (0,1) \quad f(1-\epsilon) \geq \epsilon^2/2,$$

and therefore

$$\frac{N^{N+1}}{N!}\int_0^{|w|^2} e^{-Ns}\, s^N\, ds \leq \frac{N^{N+1}}{N!}\, e^{-N}\, |w|^2\, e^{-N(1-|w|^2)^2/2}.$$

An application of Stirling's formula finishes the proof. □

As we now see this Lemma implies the localization properties of Θ_N:

Corollary 4.2. *For each $\psi \in \mathcal{B}_N$, and for each z such that $|z| < 1$,*

$$|\Theta_N^{\perp}(\psi)(z)|_s \leq \|\psi\|\, |z|\, N^{3/4}\, e^{-N(1-|z|^2)^2/4}. \tag{4.7}$$

Therefore, for all $\epsilon > 0$ there exist C, $a > 0$ such that for all $\psi \in \mathcal{B}_N$

$$\iint_{|z|<1-\epsilon} |\Theta_N^{\perp}(\psi)(z)|_s^2\, dx\, dy \leq C\, \|\psi\|^2\, N^{3/2}\, e^{-aN}. \tag{4.8}$$

Proof. By the reproducing property (4.3), we have:

$$\Theta_N^{\perp}(\psi)(z) = \langle \Theta_N^{\perp}\psi, \psi_z\rangle = \langle \psi, \Theta_N^{\perp}\varphi_z\rangle,$$

and therefore, by the Cauchy-Schwartz inequality and (4.5), for all sufficiently large N

$$|\Theta_N^{\perp}(\psi)(z)| \leq \|\psi\|\|\varphi_z\|\, |z|\, N^{1/4}\, e^{-N(1-|z|^2)^2/4}.$$

Using (4.4) we obtain (4.7), and (4.8) follows by integration. □

We will need one more general fact about the projection, Θ_N:

Lemma 4.3. *For all $z \in \mathbb{C}$ and $\psi \in \mathcal{H}_N$, one has:*

$$|(\Theta_N\psi)(z)|_s \leq (N+1) \max_{0\leq t\leq 2\pi} |\psi(e^{it}z)|_s. \tag{4.9}$$

Proof. This is a consequence of the following formula for Θ_N:

$$\left(\Theta_N\psi\right)(z) = \frac{1}{2\pi}\sum_{k=0}^{N}\int_0^{2\pi} e^{-ikt}\,\psi(e^{it}z)\, dt, \tag{4.10}$$

together with the fact that the Hermitian weight function $e^{-Nz\bar{z}/2}$ is S^1 invariant.
□

We now show how the above results can be applied to a concrete example, the sequence $P_N = \Theta_N Q \Theta_N$ where Q is the model operator:

$$Q = \frac{1}{N}\frac{d}{dz} + \mu z$$

acting on the space $\mathcal{H}_N$. The symbol of $\{P_N\}$ is the function $f = \mu z + \overline{z}$ restricted to the unit disk. Notice that $\{\Re f\,,\, \Im f\}$ is identically equal to $\mu^2 - 1$, which we will assume is negative, i.e. we now take $\mu \in (-1,1)$. Notice that for each $\lambda \in \mathbb{C}$ there exists a unique z_0 such that $\lambda = f(z_0)$.

Every complex number is an eigenvalue of the operator Q. Specifically, for each $z_0 \in \mathbb{C}$ consider the state

$$\varphi_{\mu,z_0}(z) := \sqrt{N}\, e^{-N|z_0|^2/2}\, e^{Nz\overline{z_0}}\, e^{-\frac{N}{2}\mu(z-z_0)^2}.$$

One can verify that $\varphi_{\mu,z_0} \in \mathcal{B}_N$ because $|\mu| < 1$. The state φ_{μ,z_0} is the (quantum) translate of the basic "squeezed state" at the origin, $\sqrt{N}e^{-\frac{N}{2}\mu z^2}$, to the point z_0, and its norm is a universal constant (independent of N and of z_0). It is trivial to verify that

$$Q\varphi_{\mu,z_0}(z) = f(z_0)\, \varphi_{\mu,z_0}(z).$$

As a section of $L^{\otimes N} \to \mathbb{C}$, the length of φ_{μ,z_0} at z is

$$|\varphi_{\mu,z_0}(z)|_s = \sqrt{N}\, e^{-NF(z)} \tag{4.11}$$

where F is the real quadratic function

$$F(z) = -\Re\Big(z\overline{z_0} - \frac{1}{2}\mu(z-z_0)^2\Big) + z\overline{z}/2 + |z_0|^2/2.$$

F is non-negative, vanishing exactly at $z = z_0$. The states φ_{μ,z_0} are the model of the polarized Hermite states associated with a point. Our result is as follows:

Proposition 4.4. *With the previous notation and if $\mu \in (-1,1)$ and $|z_0| < 1$,*

$$\frac{\|(P_N - \lambda)\Theta_N(\varphi_{\mu,z_0})\|}{\|\Theta_N(\varphi_{\mu,z_0})\|} = O(N^{1/4}\, e^{-aN})$$

for some $a > 0$.

Proof. We first will show that

$$\|\Theta_N(\varphi_{\mu,z_0})\|^2 = \|\varphi_{\mu,z_0}\|^2 + O(e^{-aN}). \tag{4.12}$$

As noted above, $\|\varphi_{\mu,z_0}\|$ is a constant independent of N. For simplicity, a will denote a positive constant that may not be the same at each occurrence.

Let Δ denote a disk of radius less than one containing z_0 in its interior. Then, by (4.9) and the decay properties of φ_{μ,z_0}

$$\|\Theta_N(\varphi_{\mu,z_0})\|^2 = \iint_\Delta |\Theta_N(\varphi_{\mu,z_0})|_s^2\, dx\, dy + O(e^{-aN}). \tag{4.13}$$

Next, notice that on Δ, $|\varphi_{\mu,z_0}(z)|_s$ is uniformly bounded by a constant times $\sqrt{N}$, and therefore (by (4.9) again) $|\Theta_N(\varphi_{\mu,z_0})(z)|_s$ is bounded by a constant times $N^{3/2}$ there. It follows that

$$\Big| |\Theta_N(\varphi_{\mu,z_0})(z)|_s^2 - |\varphi_{\mu,z_0}(z)|_s^2 \Big| = |\Theta_N^\perp(\varphi_{\mu,z_0})(z)|_s \cdot O(N^{3/2}),$$

with a constant uniformly on $z \in \Delta$. Therefore, by (4.8),

$$\iint_\Delta |\Theta_N(\varphi_{\mu,z_0})|_s^2 \, dx\, dy = \iint_\Delta |\varphi_{\mu,z_0}|_s^2 \, dx\, dy + O(e^{-aN}).$$

Once again, by the localization properties of φ_{μ,z_0} we have

$$\|\varphi_{\mu,z_0}\|^2 = \iint_\Delta |\varphi_{\mu,z_0}|_s^2 \, dx\, dy + O(e^{-aN}).$$

The last two equations and (4.13) imply (4.12).

Let us now turn our attention to the vector $(P_N - \lambda)\Theta_N(\varphi_{\mu,z_0})$. In the standard orthonormal basis $\{|k\rangle\}$, the matrix $T = (t_{lm})$ of P_N is tri-diagonal. Specifically, the only non-zero elements of this matrix are:

$$t_{k,k+1} = \sqrt{k/N} \quad \text{and} \quad t_{k+1,k} = \mu\sqrt{k/N}. \tag{4.14}$$

Let $\varphi_{\mu,z_0} = \sum_{k=0}^\infty a_k |k\rangle$. Since φ_{μ,z_0} is an eigenfunction of Q with eigenvalue λ,

$$(P_N - \lambda)\Theta_N(\varphi_{\mu,z_0}) = \Big(\mu\, a_{N-1} - \lambda\, a_N\Big)|N\rangle$$

We claim that both a_{N-1} and a_N are exponentially small in N. Indeed

$$a_N|N\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-iNt}\, \varphi_{\mu,z_0}(e^{it}z)\, dt,$$

which implies that for all $z \in \mathbb{C}$

$$|a_N\, z^N| \le \frac{\sqrt{N!}}{N^{(N+1)/2}} \max_t |\varphi_{\mu,z_0}(ze^{it})|,$$

where the absolute value is the standard one. Evaluating both sides at $z = 1$ and applying Stirling's formula we obtain

$$|a_N| \le CN^{-1/4}\, e^{-N/2} \max_t |\varphi_{\mu,z_0}(e^{it})|.$$

But $e^{-N/2}\max_t |\varphi_{\mu,z_0}(e^{it})| = \max_t |\varphi_{\mu,z_0}(e^{it})|_s$, and therefore

$$e^{-N/2}\max_t |\varphi_{\mu,z_0}(e^{it})| = O(\sqrt{N}e^{-aN})$$

since $|z_0| < 1$ (see 4.11 and the remarks following it). Therefore $a_N = O(N^{1/4}e^{-aN})$, and similarly for a_{N-1}. ☐

Much more generally we can start with a pseudo-mode of a general Berezin-Toeplitz operator, Q, on Bargmann space. By the localization properties of the projector Θ_N, the projection by Θ_N of the pseudomode will be a pseudomode of $\Theta_N Q \Theta_N$.

4.2. **Quantization of the torus.** We consider the standard torus, $X = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}^2$, with the complex structure arising from that of $\mathbb{C}$. A quantizing line bundle on X is holomorphically trivial when pulled-back to $\mathbb{C}$, therefore its sections can be identified with entire functions on $\mathbb{C}$ satisfying a transformation law with respect to translations by elements of Λ. It is well-known that the functions that arise in this manner are precisely theta functions. The quantizing line bundle is not unique, since one can always tensor a given one with the flat line bundles over X. This gives rise to theta functions with characteristics.

4.2.1. *The Hilbert spaces.* A quantizing line bundle over X can be constructed from a cocycle $\chi : \mathbb{C} \times \Lambda \to \mathbb{C} \setminus \{0\}$ given by:

$$\chi(z\,,\, m+in) \;=\; (-1)^{mn}\, e^{\pi[z(m-in)+\frac{1}{2}(m^2+n^2)]} e^{-2\pi i[m\mu+n\nu]}, \tag{4.15}$$

where μ and ν are fixed real numbers (the so-called characteristics of the bundle). χ is called a cocycle because it satisfies the condition

$$\chi(z,\lambda)\,\chi(z+\lambda,\mu) \;=\; \chi(z,\lambda+\mu). \tag{4.16}$$

The quantizing line bundle is the quotient of $\mathbb{C}\times\mathbb{C}$ by the equivalence relation:

$$(z,a) \sim (w,b) \quad \Leftrightarrow \quad \exists \lambda \in \Lambda \text{ such that } (w,b) = (z+\lambda\,,\, \chi(z,\lambda)a).$$

For simplicity we will only consider here theta functions with characteristics $(\mu,\nu) = (0,0)$.

We observe the following features of this construction:

(1) The sections of this line bundle, L, are naturally identified with the functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\forall (z,\lambda) \in \mathbb{C}\times\Lambda \qquad f(z+\lambda) \;=\; \chi(z,\lambda)\, f(z). \tag{4.17}$$

(Indeed the section associated to one such f is defined by:

$$s_f([z]) \;=\; [(z, f(z))]$$

where the square brackets denote equivalence classes.)

(2) For any integer N the N-th power of χ, χ^N is again a cocycle. The line bundle it defines is the N-th tensor power of L, $L^{\otimes N}$.

(3) A Hermitian structure on L is defined by a function $h : \mathbb{C} \to \mathbb{R}^+$ satisfying:

$$h(z) \;=\; |\chi(z,\lambda)|\, h(z+\lambda). \tag{4.18}$$

The Hermitian metric we will consider is: $|[(z,a)]| = |a| e^{-\pi|z|^2/2}$.

Given these data, it is easy to see that the space $\mathcal{H}_N$ of holomorphic sections of the line bundle $L^{\otimes N}$ can be identified with the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying: $\forall z \in \mathbb{C},\ m+in \in \Lambda$

$$f(z+m+in) \;=\; (-1)^{Nmn}\, e^{N\pi[z(m-in)+\frac{1}{2}(m^2+n^2)]}\, f(z).$$

Its Hilbert space structure is given by the inner product

$$< f\,,\, g > = \int_{\mathcal{F}} f(z)\,\overline{g(z)}\, e^{-N\pi|z|^2}\, dx\, dy, \tag{4.19}$$

where $\mathcal{F}$ is a fundamental domain for Λ.

The transformation law (4.17) is not the standard one for theta functions (see [2], [15]). However, if $f \in \mathcal{H}_N$, then

$$f^x(z) := e^{-N\pi z^2/2} f(z)$$

satisfies the classical transformation law

$$f^x(z+m+in) = e^{N\pi(n^2-2inz)} f^x(z).$$

For future reference we also associate to $f \in \mathcal{H}_N$ the function

$$f^y(z) := e^{N\pi z^2/2} f(z)$$

which satisfies the transformation law

$$f^y(z+m+in) = e^{N\pi(m^2+2mz)} f^y(z).$$

Note that, in particular

$$f^x(z+n) = f^x(z), \quad \text{and} \quad f^y(z+im) = f^y(z).$$

Exploiting these periodicity conditions we now exhibit two (dual) basis of $\mathcal{H}_N$.

We begin with the functions f^x. They can be expanded in Fourier series,

$$f^x(z) = \sum_{m=-\infty}^{\infty} a_m\, e(mz)$$

where we let $e(z) := e^{2\pi iz}$. The transformation law for f^x becomes a relation among the Fourier coefficients, namely $a_{m+Nn} = e^{-\pi(Nn^2+2mn)}\, a_m$. This shows that the dimension of the space of theta functions of order N is N, as the values of $a_0, \dots a_{N-1}$ determine the Fourier series. This leads to considering some special theta functions obtained by letting exactly one of the coefficients $a_0, \dots a_{N-1}$ be non-zero. These functions give rise to a basis of $\mathcal{H}_N$. More precisely:

Lemma 4.5. *For $j = 0, \dots, N-1$, let $\vartheta_j^{(N)}(z)$ be defined by the Fourier series*

$$\vartheta_j^{(N)}(z) = (2N)^{1/4}\, e^{N\pi z^2/2} \sum_{n=-\infty}^{\infty} e^{-\pi N(n+j/N)^2}\, e(z(j+Nn)).$$

Then $\vartheta_j^{(N)} \in \mathcal{H}_N$, and the set $\{\vartheta_j^{(N)},\ j = 0 \dots, N-1\}$ is an orthonormal basis of $\mathcal{H}_N$.

We can carry out a similar construction by considering the Fourier series of the functions f^y where $f \in \mathcal{H}_N$. This results in a different basis of $\mathcal{H}_N$:

Lemma 4.6. *For $k = 0, \dots, N-1$, let $\beta_k^{(N)}(z)$ be defined by the Fourier series*

$$\beta_k^{(N)}(z) = (2N)^{1/4}\, e^{-N\pi z^2/2} \sum_{m=-\infty}^{\infty} e^{-\pi N(m+k/N)^2}\, e(-iz(k+Nm)).$$

Then $\beta_k^{(N)} \in \mathcal{H}_N$, and the set $\{\beta_k^{(N)},\ k = 0 \dots, N-1\}$ is an orthonormal basis of $\mathcal{H}_N$.

It's a beautiful fact that the matrix relating these two bases is the discrete Fourier transform. We claim that

$$\beta_k^{(N)}(z) = \sum_{j=0}^{N-1} e^{-2\pi ikj/N}\, \vartheta_j^{(N)}(z). \tag{4.20}$$

We refer to [1] and especially [16], Proposition 3.17 for the reason for this.

4.2.2. *Matrix coefficients.* We now turn to a calculation of matrix coefficients of Toeplitz operators on the torus. We recall the following result of [6] (Corollary 4.8):

Lemma 4.7. *Let (x, y) be standard coordinates on the torus, so that $z = x + iy$ in the previous formulas. Let $f(x, y)$ be a symbol that is in fact a smooth periodic function of y alone. Then the $\vartheta_j^{(N)}$ are eigenvectors of the B-T operator $T_f = \{T^{(N)}\}$ with multiplier f,*

$$T^{(N)}\vartheta_j^{(N)} = \lambda_j^{(N)}\vartheta_j^{(N)}, \quad \text{where} \quad \lambda_j^{(N)} = \sum_{n=-\infty}^{\infty} a_n\, e^{-\pi n^2/2N}\, e^{-2\pi nj/N} \ \sim f|_{y=-j/N}$$

and where the a_n are the Fourier coefficients of f (with respect to y).

Notice that in particular all such operators are normal. Similarly, the basis $\beta_k^{(N)}$ consists of eigenvectors of any Toeplitz operator with total symbol a function of x alone.

Continuing our calculations of matrix coefficients, let us now take a symbol f of the form:

$$f(x, y) = h(y)\, e^{2\pi ilx}, \qquad \text{where} \quad l \in \mathbb{Z}. \tag{4.21}$$

Lemma 4.8. *If f is of the form (4.21), then $\langle f\vartheta_j^{(N)}, \vartheta_k^{(N)}\rangle$ is zero unless $k = (j+l)$ mod N, in which case*

$$\langle f\vartheta_j^{(N)}, \vartheta_{(j+l)\, mod\, N}^{(N)}\rangle = h(-\frac{2j+l}{2N}) + O(1/N).$$

More precisely, the matrix coefficient above is equal to

$$e^{-\pi l^2/2N}\,(e^{-\pi\Delta_y/2N}\, h)(-\frac{2j+l}{2N})$$

where $\Delta_y = \frac{d^2}{dy^2}$.

Proof. It is easy to verify the first statement, and, in case $k = j + l \bmod N$, one computes that

$$\langle f\vartheta_j^{(N)}, \vartheta_{j+l \bmod N}^{(N)}\rangle = \sqrt{2N} \int_0^1 \Psi_{j,l}(y)\, h(y)\, dy$$

where

$$\Psi_{j,l}(y) = e^{-\pi l^2/2N} \sum_{n=-\infty}^{\infty} e^{-2\pi N[y+n+(2j+l)/2N]^2}.$$

If $h = 1$, then we have:

$$\langle e^{2\pi i l x}\,\vartheta_j^{(N)}, \vartheta_{(j+l) \bmod N}^{(N)}\rangle = \sqrt{2N}\, e^{-\pi l^2/2N} \sum_{n=-\infty}^{\infty} \int_0^1 e^{-2\pi N[y+n+(2j+l)/2N]^2}\, dy$$

$$= \sqrt{2N}\, e^{-\pi l^2/2N} \int_{-\infty}^{\infty} e^{-2\pi N s^2}\, ds = e^{-\pi l^2/2N}. \tag{4.22}$$

This is $1 + O(1/N)$ and therefore satisfies the desired estimate. To proceed in case h is not constant, notice that by the Poisson summation formula $\Psi_{j,l}$ can be written as

$$\Psi_{j,l} = \frac{e^{-\pi l^2/2N}}{\sqrt{2N}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/2N}\, e^{2\pi i k[y+\frac{2j+l}{2N}]}$$

Therefore

$$\langle f\vartheta_j^{(N)}, \vartheta_{(j+l) \bmod N}^{(N)}\rangle = e^{-\pi l^2/2N} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/2N}\, e^{\pi i k \frac{2j+l}{N}}\, \hat{h}(-k) \sim h(-\frac{2j+l}{2N}).$$

□

Corollary 4.9. *Let $k_1 < k_2$ be two integers, and let $h_l(x)$ be smooth 1-periodic functions, $k_1 \leq l \leq k_2$. Then there exists a B-T operator on the torus, $\{T^{(N)}\}$, with principal symbol*

$$f(x, y) = \sum_{l=k_1}^{l=k_2} h_l(y)\, e^{2\pi i l x}$$

and such that for each N the matrix entries of $T^{(N)}$ in the basis $\{\vartheta_j^{(N)}\}$ are given by the formula

$$\langle T^{(N)}\vartheta_j^{(N)}, \vartheta_{(j+l)\ mod\, N}\rangle = h_j(-\frac{2j+l}{2N}) + O(N^{-\infty}), \tag{4.23}$$

where the estimate is uniform (in j, l).

Proof. For each l there exists a sequence of periodic functions $f_{l,m}$, $m = 0, 1, \ldots$ such that

$$\left(e^{-\pi l^2/2N}\, e^{-\pi\Delta_y/2N}\right) \sum_{m=0}^{\infty} N^{-m}\, f_{l,m}(y) \sim h_l(y).$$

where the left-hand side is considered a formal power series of $1/N$ (clearly $f_{l,0}(y) = h_l(y)$). By the Borel summation method, there exists an N-dependent function $f_l(y, N)$ periodic and smooth in y such that $f_l(y, N) \sim \sum_{m=0}^{\infty} N^{-m}\, f_{l,m}(y)$, estimates the C^∞ topology. By the previous Lemma, the matrix coefficients of the Toeplitz operator with N-dependent multiplier $f(x, y; N) = \sum_{l=k}^{l=m} f_l(y, N)\, e^{2\pi i l x}$ satisfy (4.23). Since the sum over l is finite the estimates are uniform. □

4.2.3. *Localization of pseudomodes.* We now proceed to describe in concrete terms the phase space localization of the pseudomodes constructed in §2. We begin by pointing out that the $\vartheta_j^{(N)}$ concentrate, as $N \to \infty$, along the circles on the torus defined by $y =$constant. This is because the ϑ_j are eigenvectors of Toeplitz operators with symbols $f = f(y)$. A precise statement is:

Lemma 4.10. *Fix $y_0 = \frac{j_0}{N_0}$ a rational number modulo 1. Then the microsupport of the sequence $\{\vartheta_{kj_0}^{(kN_0)}\ ;\ k = 1, 2, \ldots\}$ equals the circle $y = y_0$ on the torus. The $\beta_j^{(N)}$ accumulate along circles $x = x_0$ in an analogous fashion.*

An even more precise statement is that the sequences of this lemma are Legendrian states associated to the corresponding circles, in the sense of [7].

Consider now a pseudomode, ψ_N, associated with a Toeplitz operator $T^{(N)}$ on the torus, with microsupport $(x_0, y_0) \in X$. It follows from the previous lemma that the components of ψ_N

$$a_j^{(N)} := \langle \psi_N, \vartheta_j^{(N)} \rangle$$

in the basis of the ϑ_j concentrate around the values j where $j/N \approx y_0$. Similarly, the coefficients

$$b_j^{(N)} := \langle \psi_N, \beta_j^{(N)} \rangle$$

concentrate around the values j where $j/N \approx x_0$. More precisely, the concentration occurs in a neighborhood of size $O(\sqrt{N})$ of these values. As mentioned, the sequence $b_j^{(N)}$ is the finite Fourier transform of $a_j^{(N)}$. Therefore, *in these coordinates the pseudomode is localized on both sides of the Fourier transform.*

4.3. **The complex projective line.** We begin by reviewing the quantization of $\mathbb{P}^1$, for completeness. The quantization of the complex projective line arises in connection with the irreducible representations of SU(2). Recall that up to isomorphism such irreducible representations are those realized in the spaces

$$\mathcal{H}_N := \{\, f(w_1, w_2)\ ;\ f \text{ a homogeneous polynomial of degree } N\,\}.$$

Specifically, if $f \in \mathcal{H}_N$ and $g \in \mathrm{SU}(2)$, then

$$(g \cdot f)(w_1, w_2) = f(g^{-1} \cdot (w_1, w_2))$$

where the action on the right-hand side is the natural action of SU(2) on $\mathbb{C}^2$. Here $N = 0, 1, 2, \cdots$ is a non-negative integer. These representations are unitary if we put on $\mathcal{H}_N$ the Hermitian inner product

$$\langle f_1, f_2 \rangle = \int_{S^3} f_1\, \overline{f_2}\, dV_{S^3}\,,$$

where $S^3 \subset \mathbb{C}^2$ is the unit sphere and dV_{S^3} is its volume form. We note without proof that the vectors

$$|j, N\rangle = \sqrt{\frac{N+1}{\pi}}\, \sqrt{C_N^j}\, w_1^j\, w_2^{N-j}, \quad 0 \le j \le N \tag{4.24}$$

form an *orthonormal* basis of $\mathcal{H}_N$ (consisting of eigenvectors for the operator induced by $\sigma_3 \in \mathrm{su}(2)$, see below).

The circle group, $S^1 \subset \mathbb{C}$, acts freely on S^3, by complex multiplication. Therefore S^3 is a circle bundle over the abstract quotient, $X := S^3/S^1$, which can be identified with the space $\mathbb{P}^1$ of all complex lines through the origin in $\mathbb{C}^2$. There is a natural Hermitian line bundle, L^*, over $\mathbb{P}^1$ (the one whose fiber at $\ell \in \mathbb{P}^1$ is ℓ itself), and in fact $P = S^3 \subset L^*$ is the unit circle bundle. The dual bundle, $L \to \mathbb{P}^1$, is the so-called hyperplane bundle and, as it turns out, quantizes $\mathbb{P}^1$ (meaning that the natural connection on it has curvature the Fubini-Study symplectic form of $\mathbb{P}^1$). By homogeneity, the functions on $\mathcal{H}_N$ transform very simply along the orbits of S^1, and the inner product above is invariant under S^1. In fact we can regard the elements of $\mathcal{H}_N$ precisely as the holomorphic sections of $L^{\otimes N}$. We are therefore exactly in the setting of the previous sections.

The space $\mathbb{P}^1$ is isomorphic to a two-dimensional sphere, as follows. Define a map:

$$\Phi : \mathbb{P}^1 \to \mathrm{su}(2) \tag{4.25}$$

by the following rule: For each $\ell \in \mathbb{P}^1$, $\Phi(l)$ is the matrix having ℓ as an eigenspace, with associated eigenvalue $i/2$, and $\ell^\perp$ as another eigenspace, with associated eigenvalue $-i/2$. (The choice of spectrum is dictated by the normalization that the area of S^3/S^2 agrees with the one induced by the Killing form.) One can show that, if we write $\ell = [w_1, w_2] \in \mathbb{P}^1$, where $(w_1, w_2) \in S^3$ is a representative, then the previous map is:

$$\Phi([w_1, w_2]) = \frac{i}{2} \begin{pmatrix} |w_1|^2 - |w_2|^2 & 2w_1\overline{w_2} \\ 2w_2\overline{w_1} & |w_2|^2 - |w_1|^2 \end{pmatrix}. \tag{4.26}$$

It is a fact that Φ is a moment map for the natural $\mathrm{SU}(2)$ action on $\mathbb{P}^1$. Recall also that the matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.27}$$

form a standard orthogonal basis of $\mathrm{su}(2)$ (such that $[\sigma_1, \sigma_2] = \sigma_3$, etc., and $\|\sigma_j\| = 1/2$) if we give to $\mathrm{su}(2)$ the SU-invariant inner product

$$\langle A, B \rangle = -\frac{1}{2} \operatorname{Tr} AB.$$

Clearly Φ is an equivariant diffeomorphism onto its image, which, geometrically, is the sphere of radius $1/2$ and, algebraically, a (co)adjoint orbit. (It turns out that the symplectic form on $\mathbb{P}^1$ is twice the area form. In general the symplectic form on an orbit of radius s is the area form divided by s, see [19] pg. 54.) We will henceforth identify $\mathbb{P}^1$ with this sphere/co-adjoint orbit.

Let $x_j : \mathrm{su}(2) \to \mathbb{R}$ be the j-th coordinate function, $x_j(A) = \langle A, 2\sigma_j \rangle$, so that the image of Φ is the sphere $\sum_{j=1}^3 x_j^2 = \frac{1}{4}$. The description of $\mathbb{P}^1$ as a sphere means that we can speak of restrictions of linear functions from $\mathrm{su}(2)$ to $\mathbb{P}^1$. In particular, a crucial role in what follows will be played by the function $I : \mathbb{P}^1 \to [0,1]$ given by

$$I(\ell) = \langle \Phi(\ell), 2\sigma_3 \rangle + \frac{1}{2}. \tag{4.28}$$

The Hamilton flow of this function with respect to the natural symplectic structure on $\mathbb{P}^1$ (as a co-adjoint orbit) is given by the action of the one-parameter subgroup $\exp(2t\sigma_3)$, and geometrically is rotation around the σ_3 axis. Accordingly, we introduce the polar angle, θ, regarded as a multivalued function on $\mathbb{P}^1$ (undefined at the poles). (I, θ) are action-angle coordinates on $\mathbb{P}^1$, and, in particular, the symplectic form on $\mathbb{P}^1$ is

$$\omega = dI \wedge d\theta.$$

Any Berezin-Toeplitz operator gives rise to a sequence of matrices, namely, the matrices representing the operator in the canonical basis, $\{|j, N\rangle\}$, of $\mathcal{H}_N$. We will compute below (approximately) the matrix of a Toeplitz operator with principal symbol a given function $f : \mathbb{P}^1 \to \mathbb{C}$. We begin with some remarks about such functions. Let

$$f(I, \theta) = \sum_{l=-\infty}^{\infty} e^{il\theta} f_l(I) \tag{4.29}$$

be the Fourier series expansion of f with respect to the action of S^1 by rotations around the σ_3 axis. Although we are writing this expansion in action-angle variables, each summand is a smooth function on the sphere, which imposes boundary conditions on the f_l. Specifically, one can prove that the functions $f_l(I)$ such that $f_l(I)e^{il\theta}$ is smooth as a function on $\mathbb{P}^1$ are of the form

$$f_l(x) = (x(1-x))^{l/2} g_l(x) \tag{4.30}$$

where g_l has a smooth extension to a neighborhood of $[0,1]$. In fact, $x_1 + ix_2 = (\frac{1}{4} - x_3^2)^{1/2} e^{i\theta}$, so

$$(I(1-I))^{l/2} e^{il\theta} = (x_1 + ix_2)^l.$$

Therefore, if (4.30) holds,

$$f_l(I)e^{il\theta} = g_l(I)(x_1 + ix_2)^l$$

which is clearly smooth on the sphere if $g_l \in C^\infty[0,1]$.

Let us now turn to the computation of matrix elements of B-T operators on $\mathbb{P}^1$. By linearity of Toeplitz quantization, it will suffice to compute the matrix of a Toeplitz operator with symbol $e^{il\theta} f_l(I)$ for a given integer l. We begin with the case $l = 0$.

Lemma 4.11. *Let α be a smooth function on $[0,1]$. Then there is a B-T operator on the sphere, $\{A^{(N)}\}$, with principal symbol $\alpha \circ I$, which is diagonal in the standard basis of $\mathcal{H}_N$ and whose j-th diagonal entry is $\alpha(j/(1+N))$, $0 \le j \le N$.*

Proof. The sequence of operators, $Z = \{Z^{(N)} : \mathcal{H}_N \to \mathcal{H}_N\}$ such that

$$Z^{(N)}|j,N\rangle = \frac{j}{N+1}\,|j,N\rangle, \qquad 0 \le j \le N$$

(where $|j,N\rangle$ is defined in (4.24)) is a Berezin-Toeplitz operator with symbol I (see Lemma 3.4 and the ensuing discussion in [6]). Let

$$A^{(N)} = \frac{1}{2\pi}\int e^{itZ^{(N)}}\,\hat{\alpha}(t)\,dt,$$

where $\hat{\alpha}$ is the Fourier transform of a compactly-supported smooth extension of α. Since Z is a self-adjoint Toeplitz operator of order zero, $\{e^{itZ^{(N)}}\}$ is a unitary Toeplitz operator of order zero (see for example Proposition 12 of [8]) and symbol e^{itx_3}. It is easy to check that $\{A^{(N)}\}$ has the desired properties. $\square$

To compute the matrix elements of operators with a smooth symbol $e^{il\theta}\,f_l(I)$ where $l \neq 0$, we introduce the raising and lowering operators. Let

$$J_k^N := i \times \text{the operator induced by } \sigma_k \text{ on } \mathcal{H}_N,\ \ k = 1,2,3.$$

Then:

$$J_1^N = \frac{1}{2}\Big(w_2\frac{\partial}{\partial w_1} + w_1\frac{\partial}{\partial w_2}\Big) \quad J_2^N = \frac{i}{2}\Big(w_2\frac{\partial}{\partial w_1} - w_1\frac{\partial}{\partial w_2}\Big)$$

and

$$J_3^N = \frac{1}{2}\Big(w_1\frac{\partial}{\partial w_1} - w_2\frac{\partial}{\partial w_2}\Big).$$

In particular, the lowering and raising operators, $J_\pm^N = J_1^N \pm iJ_2^N$, are:

$$J_+^N = w_1\frac{\partial}{\partial w_2}, \qquad J_-^N = w_2\frac{\partial}{\partial w_1}$$

(and the vectors $|j\rangle$, $j = 0,\dots,N$ are eigenvectors of J_3^N, with eigenvalue $j - \frac{N-1}{2}$.)

Lemma 4.12. *The matrix of J_-^N, resp. J_+^N, in the standard basis has zero entries except along the supra-diagonal, resp. infra-diagonal, along which the entries are equal to*

$$m_j = \sqrt{j(N-j+1)}, \qquad j = 1,\dots,N.$$

Moreover, the sequence $\{\frac{1}{N}J_\pm^N\}$ is a B-T operator with symbol $x_1 \pm ix_2$.

The first two statements follow a simple calculation; for the last statement we refer to [6].

We are now in a position to describe the matrices of the B-T operators on the sphere:

Proposition 4.13. *Let $f : \mathbb{P}^1 \to \mathbb{C}$ be a smooth function with a finite Fourier series, (4.29), in action-angle variables, and let $f_l(x) = (x(1-x))^{l/2}\,g_l(x)$ with $g_l \in C^\infty[0,1]$. Let $\mathcal{M}_\pm^{(N)}$ be the matrix of $J_\pm^N$, described in the previous Lemma.*

Then there exists a B-T operator, $\{T^{(N)}\}$, with symbol f and such that the sequence of matrices $\{\mathcal{T}^{(N)}\}$ of $\{T^{(N)}\}$ in the standard basis of $\mathcal{H}_N$ satisfies:

$$\mathcal{T}^{(N)} = \sum_l \left(\mathcal{M}^{(N)}_{sgn(l)}\right)^{|l|} \mathcal{A}^{(N)}(g_l) + O(N^{-\infty}) \tag{4.31}$$

where $\mathcal{A}^{(N)}(g_l)$ is the $(N+1)\times(N+1)$ diagonal matrix with diagonal entries $g_l(j/(N+1))$ and the estimate is in any matrix norm.

Proof. By linearity and the assumption that the sum (4.29) is finite it suffices to prove the proposition for f of the form $f(I,\theta) = e^{il\theta} f_l(I)$. The case $l=0$ is covered by Lemma 4.11. It suffices to consider the case $l>0$. Applying Lemma 4.11 to g_l we obtain a diagonal B-T operator, $\{A^{(N)}(g_l)\}$, with diagonal entries $g_l((j-1)/N)$, $0 \le j \le N$. It is clear that the B-T operator

$$T^{(N)} := \left(\frac{1}{N} J_-^{(N)}\right)^l \circ A^{(N)}$$

has the desired properties. Notice that one can choose any order of the products appearing in (4.31) and still find a B-T operator with the desired properties. □

4.3.1. *Linear Hamiltonians.* Recall the moment map, $\Phi : \mathbb{P}^1 \to \mathrm{su}(2)$ given by (4.26). Given $M \in \mathrm{sl}(2,\mathbb{C})$, we can pull-back by Φ the complex-valued linear function on su(2),

$$\mathrm{su}(2) \ni A \mapsto -\frac{1}{2}\,\mathrm{Tr}(AM).$$

Let us denote the pull-back by $F_M : \mathbb{P}^1 \to \mathbb{C}$; specifically

$$\forall \ell \in \mathbb{P}^1 \qquad F_M(\ell) = -\frac{1}{2}\,\mathrm{Tr}(\Phi(\ell)M).$$

We will call functions such as F_M *linear Hamiltonians.* Since Φ is a moment map, the assignment $M \to F_M$ is a Lie algebra morphism:

$$\forall M_1,\ M_2 \in \mathrm{sl}(2,\mathbb{C}) \qquad F_{[M_1,M_2]} = \{F_{M_1}\,,\,F_{M_2}\}.$$

In particular, if we continue to denote by $x_j : \mathbb{P}^1 \to \mathbb{R}$ the restriction to $\mathbb{P}^1$ of the coordinate functions, then we have the identity: $\{x_1\,,\,x_2\} = x_3$, and also its cyclic permutations.

If M is semi-simple, there exists $g \in \mathrm{SU}(2)$ such that gMg^{-1} is diagonal, i.e.

$$\exists g \in \mathrm{SU}(2),\ \mu \in \mathbb{C}\setminus 0 \qquad gMg^{-1} = \mu\,\sigma_3.$$

Here's a very concrete example. Take

$$V = \exp(it\sigma_2) = \begin{pmatrix} \cosh(t/2) & -i\sinh(t/2) \\ i\sinh(t/2) & \cosh(t/2) \end{pmatrix}$$

and consider $A = A(t) \in \mathrm{sl}(2,\mathbb{C})$ equal to

$$A = V\sigma_1 V^{-1} = i\sinh(t)\,\sigma_1 + \cosh(t)\,\sigma_3.$$

The classical Hamiltonian, $F_A : \mathbb{P}^1 \to \mathbb{C}$ is $F_A = i \sinh(t)\, x_1 + \cosh(t)\, x_3$, and its image is the interior of an ellipse,

$$\frac{x^2}{\cosh(t)^2} + \frac{y^2}{\sinh(t)^2} \leq \frac{1}{4}. \tag{4.32}$$

If we let $T^{(N)} : \mathcal{H}_N \to \mathcal{H}_N$ be $1/N$ times the operator image of $\frac{1}{i}A$ by the representation ρ_N, then, for all t, $T^{(N)}$ is diagonalizable with real spectrum $\{\frac{j}{N} - \frac{N-1}{2N}\ ;\ j = 0, \ldots, N-1\}$, which is contained in the major axis of the image of F_A.

Notice that $\{\Re F_A\,,\, \Im F_A\} = \cosh(t)\sinh(t)\, x_2$, and therefore for every point in the image of F_A there is exactly one x where this Poisson bracket is negative. Therefore, for each λ in the interior of the image the norm of the resolvent $\|(T^{(N)} - \lambda I)^{-1}\|$ is $O(N^\infty)$.

The level sets of the norm of the resolvent for this example resemble the equipotential curves of a uniform electric charge distribution on the line segment $[-\frac{1}{2}, \frac{1}{2}]$. Notice, incidentally, that the image under F_A of the level curves of the Poisson bracket $\{\Re F_A\,,\, \Im F_A\}$ are ellipses crossing this line segment. *This example therefore shows that there is not a direct relationship between the norm of the resolvent and the size of the Poisson bracket of the real and imaginary parts of the symbol.*

Finally, notice that every λ on the boundary of the elliptical region (4.32) satisfies the hypotheses of Theorem 1.2. The set $f^{-1}(\lambda)$ consists of exactly one point, where $\{\Re F_A\,,\, \Im F_B\} = \cosh(t)\sinh(t) x_2 = 0$. However, one of the double brackets involving real and imaginary parts of F_A is non-zero at this point. Thus Theorem 1.2 applies, with $k = 2$. Notice that, in spite of the estimates (1.9) the distance from λ to spectrum is $O(1)$. This is in agreement with Theorem 3 of [10]: Thinking of $\mathbb{P}^1$ as a *real* manifold $\mathcal{O}$, the symbol f has an obvious holomorphic extension to a complexification of $\mathcal{O}$, namely, the (co)adjoint orbit of $\mathrm{SL}(2, \mathbb{C})$ through σ_3.

5. Final Remarks

5.1. **On the numerical range.** Let $T_f = \{T^{(N)}\}$ be a Berezin-Toeplitz operator with symbol $f : X \to \mathbb{C}$. For each N, the numerical range of $T^{(N)}$ is the set

$$W_N := \{\, \langle T^{(N)}\psi, \psi\rangle\ ;\ \psi \in \mathcal{H}_N\,,\ \|\psi\| = 1\,\}. \tag{5.1}$$

We define W_∞ as the limit of the ranges W_N as $N \to \infty$:

Definition 5.1. A complex number λ is in W_∞ iff for all $\epsilon > 0$ there exists $K > 0$ such that for all $N > K$

$$\Delta_\epsilon(\lambda) \cap W_N \neq \emptyset,$$

where $\Delta_\epsilon(\lambda)$ is the disc of radius ϵ centered at λ.

Proposition 5.2. *W_∞ is the convex hull of the image of the classical symbol, f.*

Proof. It is well-known that, for each N, W_N is convex (see [12]). It follows easily that W_∞ is convex as well. Moreover, if $x \in X$ and ψ_x^N is a coherent state at x, then

$$\frac{\langle T^{(N)}\psi_x^N, \psi_x^N\rangle}{\langle \psi_x^N, \psi_x^N\rangle} \to f(x).$$

This shows that W_∞ contains the image of f.

To show that W_∞ is actually the convex hull, consider a line of equation $ax+by=c$, and let $\lambda \in W_\infty$. By definition, there exists a sequence $\lambda_N \in W_N$ converging to λ, and therefore there exists a sequence of unit vectors $\{\psi_N \in \mathcal{H}_N\}$ such that

$$\lambda_N = \langle T^{(N)}\psi_N, \psi_N\rangle \to \lambda.$$

Let us write $\lambda_N = x_N + iy_N$ for the real and imaginary parts of λ_N, and $f = p_1 + ip_2$ for the real and imaginary parts of the symbol f. Then

$$ax_N + by_N = \int_X (ap_1 + bp_2)\, |\psi_N|_s^2\, dm + O(1/N).$$

Assume that the region: $ax + by > c$ does not intersect the image of f. Then for all $\in X$ $a\phi(m) + bg(m) \leq c$, and therefore

$$ax_N + by_N \leq c \int_X |\psi_N|^2\, dm_N + O(1/N) = c + O(1/N).$$

Letting $N \to \infty$, we obtain that $ax_\infty + by_\infty \leq c$, where $\lambda = x_\infty + iy_\infty$. Thus λ, and therefore all of W_∞, is on the same side of the line $ax + by = c$ as the image of f. $\square$

5.2. **The weak Szegö limit theorem.** In the non-selfadjoint case, one has the following version of the Szegö limit theorem for B-T operators:

Proposition 5.3. *Let $T_f = \{T^{(N)}\}$ be a B-T operator with principal symbol $f : X \to \mathbb{C}$, and let $F(z)$ be a function of a complex variable analytic on a simply-connected region containing the image of f. Then*

$$\frac{1}{\dim \mathcal{H}_N} \operatorname{Tr} F(T^{(N)}) = \frac{1}{Vol\ X} \int_X F \circ f\, dm + O(1/N), \tag{5.2}$$

where dm is the Liouville measure of X.

Proof. The idea of the proof is standard; we include some details for completeness and to verify that the usual proof is valid in the current setting. As in the theory of pseudodifferential operators, for each λ not in the image of f one can construct a B-T operator, B_λ, such that the Schwartz kernel of

$$R_\lambda^{(N)} := B_\lambda^{(N)} \circ (T^{(N)} - \lambda) - I_{\mathcal{H}_N}$$

is a smooth section of the bundle $\operatorname{Hom}(L^N, L^N) \to X$ which is rapidly decreasing in N (together with all its derivatives). The principal symbol of $\{B_\lambda^{(N)}\}$ is $(f - \lambda)^{-1}$. Multiplying on the right by $(T^{(N)} - \lambda)^{-1}$ we obtain

$$(T^{(N)} - \lambda)^{-1} = B_\lambda + S_\lambda^{(N)}$$

where $\{S^{(N)}\}$ has the same properties as $R_\lambda^{(N)}$. Let Γ be a simple closed curve, positively oriented, contained in a region where $F(z)$ is analytic and containing the image of f. Then

$$F(T^{(N)}) = \frac{1}{2\pi i} \oint_\Gamma F(\lambda)\, B_\lambda^{(N)}\, d\lambda + \frac{1}{2\pi i} \oint_\Gamma F(\lambda)\, S_\lambda^{(N)}\, d\lambda.$$

The trace of the second term on the right-hand side is $O(N^{-\infty})$, while

$$\frac{1}{\dim \mathcal{H}_N} \operatorname{Tr} \oint_\Gamma F(\lambda)\, B_\lambda^{(N)}\, d\lambda = \frac{1}{\dim \mathcal{H}_N} \oint_\Gamma F(\lambda)\ \operatorname{Tr}\, B_\lambda^{(N)}\, d\lambda.$$

But it is known that $\frac{1}{\dim \mathcal{H}_N} \operatorname{Tr}\, B_\lambda^{(N)} = \frac{1}{\mathrm{Vol}\ X} \int_X (f-\lambda)^{-1}\, dm + O(1/N)$, where the estimate is uniform for λ on compact sets away from the image of f. $\square$

In particular, if λ is a complex number away from the image of f, one has:

$$\frac{1}{\dim \mathcal{H}_N} \operatorname{Tr}[(T^{(N)} - \lambda)^{-1}] = \frac{1}{\mathrm{Vol}\ X} \int_X \frac{1}{f-\lambda}\, dm + O(1/N). \tag{5.3}$$

Clearly the left-hand side of this equation is a sequence of analytic functions in λ defined away from the union of the spectra of the $T^{(N)}$. On the other hand, the integral $\int_X \frac{1}{f-\lambda}\, dm$ is analytic away from the image of f. Examples show that the spectral radius of $T^{(N)}$ has a limit, R, such that the image of f is *not* contained in the circle of radius R. It is not immediate to extend (5.3) to λ with $|\lambda| > R$ but inside the image of f.

Appendix A. Hermite distributions and symbol calculus

Oscillatory integrals and symbols

We place ourselves in the setting of §2.2: Let $\mathcal{R} \subset \mathcal{Z}$ be a closed conic isotropic submanifold, and $Y \hookrightarrow X$ the reduced isotropic submanifold of X. Hermite distributions in $I^m(P, \mathcal{R})$ are defined locally as oscillatory integrals. To write down an explicit form for these integrals, we'll choose Darboux coordinates $(q,p) \in \mathbb{R}^{2n}$ for X with $q = (q', q'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}$ such that $Y = \{q'' = p = 0\}$. Here $l = \dim Y$. For P we then have coordinates $z = (q, p, \theta)$, and we can always find a function $h(p,q)$ such that $\alpha|_{p=0} = d\theta - dh|_{p=0}$. The lift of Y to the isotropic $\mathcal{R}$ is given by specifying that $\theta = h(q', 0)$.

Now we'll introduce phase coordinates $(\tau, \eta_1, \eta_2) \in \mathbb{R}_+ \times \mathbb{R}^{n-l} \times \mathbb{R}^n$, and a phase function

$$\phi(z, \tau, \eta) = \tau(\theta - h(q,p)) + \eta_1 \cdot q'' + \eta_2 \cdot p,$$

which parametrizes $\mathcal{R}$. A distribution $u \in I^m(P, \mathcal{R})$ can be written locally as

$$u(z) = \int e^{i\phi(z,\tau,\eta)} a(x, \tau, \eta/\sqrt{\tau})\, d\tau\, d\eta,$$

where the amplitude $a(z,\tau,u)$ is rapidly decreasing in u and has a expansion in τ of the form

$$a(z,\tau,\eta) \sim \tau^{m-1/2} \sum_{j=0}^{\infty} \tau^{-j/2} a_j(z,\eta).$$

The symplectic spinor symbol $\sigma(u)$ should be thought of as a_0 written in a suitably invariant way. The choice of ϕ defines for each $\rho \in \mathcal{R}$ a canonical isomorphism between $T^*(\mathbb{R}^{n-l} \times \mathbb{R}^n)$ and the symplectic normal N_ρ, by which the $H_\infty(N_\rho)$ portion of the symbol may be pulled back to a rapidly decreasing function of η. This gives the η dependence of a_0, while the half-form portion of the symbol encodes the dependence on $z \in P$ in a coordinate-independent way.

The construction of u in §3.1 gives u such that $\sigma_u = \nu_u \otimes \kappa_u \otimes e$. Here e is a Gaussian in $H_\infty(Z_\rho^\circ)$ which can be explicitly computed in terms of the metric. The component $\kappa_u \in H_\infty(E_\rho)$ satisfies $\mathcal{L}(\kappa_u) = 0$ according to the construction. This does not fix κ_u, but we are free to assume that κ_u is also Gaussian. Here $\mathcal{L}$ is the operator on $H_\infty(E_\rho)$ given by the Heisenberg representation of the Hamiltonian vector field ξ of σ_Q. But according to Theorem 11.4 of [5], $\xi_{\mathcal{Z}}$ is just the lift of the Hamiltonian vector field Ξ_f on X up to $\mathcal{Z}$. So $\mathcal{L}$ could be written explicitly in terms of f.

By this construction of u, the leading amplitude $a_0(z,\eta)$ is a Gaussian function of η:

$$a_0(z,\eta) = c(z) e^{-\eta^t A^{-1} \eta/2}$$

where A is a symmetric matrix with positive definite real part. Arguing as in the proof of Theorem 3.12 in [7], we can write $u_N = \Pi_N(u)$ as

$$u_N((q,p,h(q,p)) = e^{iNh(q,p)} \int e^{-iN\theta} e^{i\phi} a(z,\tau,\eta/\sqrt{\tau})\, d\theta\, d\tau\, d\eta,$$

(near $q'' = p = 0$) and apply stationary phase to the $N \to \infty$ limit. The asymptotic result is that

$$u_N(q,p,h(q,p)) \sim c(q') N^{n-l/2-1/2} e^{-N(q'',p)A(q'',p)^t/2},$$

where $c(q')$ is independent of N. In particular, at a point $z_0 \in P$ lying above Y, we have

$$u_N(z_0) \sim c(z_0) N^{n-l/2-1/2}. \tag{A.1}$$

A review of the Hermite calculus

The composition of Hermite distributions is described in Theorem 9.4 of [5], which implies that

$$\Pi : I^m(R,\mathcal{R}) \to I^m(R,\mathcal{R}).$$

The symbol calculus corresponding to this composition is based on symplectic linear algebra found in §6 of [5], where the reader can find full details. Since the symbol calculus is somewhat involved, we begin with a review of the general details of the symbol map before applying it to our case.

Let V and W be symplectic vector spaces, $\Gamma \subset V \times W^-$ a Lagrangian subspace and $\Sigma \subset W$ an isotropic subspace. We think of Γ as a canonical relation from W to V; $\Gamma \circ \Sigma$ is an isotropic subspace of V. We will make the simplifying assumption that

$$U_0 = \{ w \in \Sigma ;\, (0,w) \in \Gamma \} = 0, \tag{A.2}$$

valid in the applications of the calculus to this paper.

We assume given a symplectic spinor on Σ and a half-form on Γ. Recall that if $H_\infty(V)$ denotes the space of C^∞ vectors in the metaplectic representation of the metaplectic group of the symplectic vector space V, the space of symplectic spinors on Σ is $H_\infty(\Sigma^\circ/\Sigma) \otimes \bigwedge^{1/2}(\Sigma)$.

Under the assumption (A.2), the (linear) symbol map of the Hermite calculus is a linear map

$$H_\infty(\Sigma^\circ/\Sigma) \otimes \textstyle\bigwedge^{1/2}(\Sigma) \otimes \bigwedge^{1/2}(\Gamma) \to H_\infty((\Gamma\circ\Sigma)^\circ/\Gamma\circ\Sigma) \otimes \bigwedge^{1/2}(\Gamma\circ\Sigma). \tag{A.3}$$

Our first goal here is to describe the map (A.3). There are two ingredients in its construction, which will be examined separately. First however we must introduce the following vector spaces:

Definition A.1. $U_1 := \{ w \in \Sigma^\circ ;\, (0,w) \in \Gamma \} \subset W$, and

$$U := \text{image of } U_1 \text{ in } \Sigma^\circ/\Sigma \cong U_1.$$

These spaces enter the calculus in the following way:

Lemma A.2. *The subspace*

$$U \subset \Sigma^\circ/\Sigma$$

is isotropic, and there is a natural identification

$$U^\circ/U \cong (\Gamma\circ\Sigma)^\circ/\Gamma\circ\Sigma\,. \tag{A.4}$$

The first ingredient in the symbol map is a canonical isomorphism:

Lemma A.3. *Under (A.2), there exists a canonical isomorphism*

$$\textstyle\bigwedge^{1/2}(\Sigma)\otimes\bigwedge^{1/2}(\Gamma) \;\cong\; \bigwedge^{1/2}(U_1^\circ)\otimes\bigwedge^{1/2}(\Gamma\circ\Sigma)\,.$$

Proof. Let

$$\rho : \Gamma \oplus \Sigma \to U_1^\circ \tag{A.5}$$

be the map $\rho((v,w),w_1) = w - w_1$. One can show that the image of this map is exactly U_1°. Moreover, because of (A.2), the projection $((v,w),w_1) \mapsto v$ is an isomorphism

$$\ker(\rho) \cong \Gamma\circ\Sigma.$$

This is the non-trivial vertical arrow in the diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \ker(\rho) \arrow[r] \arrow[d] & \Gamma \oplus \Sigma \arrow[r] & U_1^\circ \arrow[r] & 0 \\
 & \Gamma \circ \Sigma \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
\tag{A.6}
$$

The horizontal sequence is just the natural short exact sequence associated to the surjection ρ.

Having established the existence of these exact sequences the desired isomorphism follows from the behavior of the functor $\bigwedge^{1/2}$ when applied to short exact sequences and to direct sums. □

The second ingredient in the Hermite calculus is the following:

Lemma A.4. *Under the assumption (A.2), there exists a canonical map*

$$\mathcal{S}(\Sigma^\circ/\Sigma) \to \overline{\bigwedge}^{-1/2}(U_1) \otimes \mathcal{S}((\Gamma \circ \Sigma)^\circ/\Gamma \circ \Sigma)\,.$$

Proof. This is based on Lemma (A.2) and the following generalization of a map defined by Kostant:

Claim: Let A (in our case we will take $A = \Sigma^\circ/\Sigma$) be a symplectic vector space, and $U \subset A$ an isotropic subspace. Then there is a natural map

$$H_\infty(A) \to \overline{\bigwedge}^{-1/2}(U) \otimes H_\infty(U^\circ/U)\,. \tag{A.7}$$

The desired map follows from these two claims, if we recall that $U = U_1$ (because of (A.2). □

To obtain the symbol map (A.3), tensor the maps from the lemmas and use the fact that the symplectic form on W defines a natural identification

$$\overline{\bigwedge}^{-1/2}(U_1) \otimes \bigwedge^{1/2}(U_1^\circ) \cong \mathbb{C}.$$

Proof of Proposition 2.5.

Given $v \in I^m(P, \mathcal{R})$, we want to calculate the symbol of $u = \Pi(v) \in I^m_\Pi(R, \mathcal{R})$. In order to apply the symbol calculus reviewed above, we need to rewrite $\Pi(v)$ as the composition of a Lagrangian distribution with a Hermite. Thus we introduce $\pi : P \times P \to P$, the projection through the left factor, and $F : P \times P \to P \times P \times P$ the map $F(p_1, p_2) = (p_1, p_2, p_2)$. We can then write

$$\Pi(v) = \pi_* F^*(\Pi \boxtimes u),$$

where (abusing notation slightly) Π here denotes the integral kernel.

For the symbol calculation it suffices to localize to $\rho \in \mathcal{R} \subset T^*P$. To simplify notation we will introduce vector spaces

$$V = T_\rho(T^*P), \qquad W = V \times V \times V.$$

In W two copies of V carry the opposite symplectic form, but to simplify notation we just denote the vector space. We define the vector spaces R_ρ, Z_ρ, E_ρ, etc. as in §3.1. Note that Z_ρ is a symplectic subspace of V, and R_ρ is isotropic in Z_ρ. Locally, the symbol of $v \in I(P, \mathbb{R})$ can be written

$$\begin{aligned} \sigma(v)|_\rho &= \nu \otimes \kappa \otimes \lambda \\ &\in \bigwedge\nolimits^{1/2}(R_\rho) \otimes H_\infty(E_\rho) \otimes H_\infty(Z_\rho^\circ). \end{aligned}$$

As an integral kernel, $\Pi \in I(P \times P, \mathcal{Z}_\rho^\Delta)$ with symbol

$$\sigma(\Pi) = \sqrt{dz} \otimes e \otimes \bar{e} \in \bigwedge\nolimits^{1/2}(Z_\rho) \otimes H_\infty(Z_\rho^\circ) \otimes H_\infty(Z_\rho^\circ),$$

where dz is the canonical volume form given by the symplectic form on Z_ρ.

The operator $\pi_* F^*$ is a Lagrangian FIO with canonical relation

$$\Gamma = \{(v; v, w, w);\ v, w \in V\} \subset V \times W.$$

(To see that it's Lagrangian one needs to keep track of the signs of the symplectic forms). The combination $\Pi \boxtimes u$ is a Hermite distribution associated to the isotropic

$$\Sigma = \{(z, z, y);\ z \in Z_\rho, y \in R_\rho\}.$$

Note that $\Gamma \circ \Sigma = R_\rho$.

First we'll describe the spaces that play a role in Lemmas A1–4. To begin, note that R_ρ° was calculated in Lemma 2.3 to be $Z_\rho^\circ \oplus E_\rho$, where E_ρ is the symplectic normal of R_ρ as a subspace of Z_ρ. Then we see that

$$\Sigma^\circ/\Sigma = Z_\rho^\circ \times Z_\rho^\circ \times (Z_\rho^\circ \oplus E_\rho).$$

Then

$$U_1 = \{(0, v, v);\ v \in Z_\rho^\circ\} \subset W,$$

and U is the same set as a subspace of Σ°/Σ, so that $U^\circ/U \cong Z_\rho^\circ \oplus E_\rho = R_\rho^\circ/R_\rho$.

Introducing the map ρ as above, we see that

$$\mathrm{Image}(\rho) = U_1^\circ = U_1 \oplus (V \times Z_\rho \times Z_\rho) \subset W,$$

and $\ker \rho \cong R_\rho$.

With these identifications, we can decompose the pieces in the symbol map (A.2). First of all, in Lemma A.3 the map reduces to the identity map on

$$\bigwedge\nolimits^{1/2}(Z_\rho) \otimes \bigwedge\nolimits^{1/2}(R_\rho) \otimes \bigwedge\nolimits^{1/2}(V),$$

combined with the obvious decomposition

$$\bigwedge\nolimits^{1/2}(V) \cong \bigwedge\nolimits^{1/2}(Z_\rho) \otimes \bigwedge\nolimits^{1/2}(Z_\rho^\circ).$$

Also note that since $\Sigma^\circ/\Sigma = Z_\rho \times Z_\rho \times (Z_\rho \oplus E_\rho)$ we have

$$H_\infty(\Sigma^\circ/\Sigma) = H_\infty(Z_\rho^\circ) \otimes H_\infty(Z_\rho^\circ) \otimes H_\infty(Z_\rho^\circ) \otimes H_\infty(E_\rho).$$

This means we can break the symbol map (A.2) into three pieces. The first is the identity map

$$H_\infty(E_\rho) \otimes \bigwedge^{1/2} R_\rho \to H_\infty(E_\rho) \otimes \bigwedge^{1/2} R_\rho.$$

The second is a canonical pairing (A.6) applied to the symplectic space $Z_\rho^\circ \times Z_\rho^\circ$ with the diagonal as Lagrangian subspace:

$$H_\infty(Z_\rho^\circ) \otimes H_\infty(Z_\rho^\circ) \cong \overline{\bigwedge}^{-1/2}(Z_\rho^\circ).$$

Finally, the third is the natural identification

$$\overline{\bigwedge}^{-1/2}(Z_\rho^\circ) \otimes \bigwedge^{1/2}(Z_\rho^\circ) \otimes \bigwedge^{1/2}(Z_\rho) \otimes \bigwedge^{1/2}(V) \cong \mathbb{C},$$

defined by the symplectic forms.

Applying this to the symbol of $\Pi(u)$, the third piece shows some natural half-forms canceling, the second gives contributes a factor $\langle e, \lambda \rangle$ which can be absorbed by changing the half-form component $\nu \mapsto \nu'$. And the first map then gives the stated conclusion:

$$\sigma(\Pi(v)) = \nu' \otimes \kappa \otimes e.$$

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