NOTES ON QUANTUM MECHANICS

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Contents

1. The Basic Setup

1.1. **The Axioms of non-relativistic quantum mechanics.**

1. The states of a quantum-mechanical system are in bijective correspondence with the complex lines in a Hilbert space, $\mathcal{H}$. If $\psi \in \mathcal{H} \setminus 0$, the line through ψ will be denoted: $[\psi]$.
2. The quantum observables are self-adjoint operators in $\mathcal{H}$.
3. If the system is in the state $[\psi]$, the expected value of an observable A in the state $[\psi]$ is
$$\frac{\langle A\psi, \psi\rangle}{\langle \psi, \psi\rangle}.$$
More precisely, let $\{dP_\lambda\}$ be the spectral resolution of A. Then the probability measure for the result of the observation of A on $[\psi]$ is
$$\frac{\langle dP_\lambda\psi, \psi\rangle}{\langle \psi, \psi\rangle} d\lambda$$
4. If $\hat{H}$ is the quantum Hamiltonian (total energy observable), and if the system is in the state $[\psi_0]$ at time $t = 0$, then the system at time t will be in the state $[\psi_t]$ where
$$i\hbar \frac{\partial \psi_t}{\partial t} = \hat{H}(\psi_t).$$
This is the Scrhödinger equation.

A consequence of 4 is that the quantum analogue of the Poisson bracket is:
$$[A\,,\,B]_{PB} := \frac{i}{\hbar}(AB - BA).$$

1.2. **The Harmonic Oscillator.** On $L^2(\mathbb{R}^n)$:

$$Q = \text{multiplication by } x, \qquad P = \frac{\hbar}{i}\frac{\partial}{\partial x}. \tag{1.1}$$

Canonical commutation relations: (CCR's):

$$[P_j, Q_k] = \delta_{j,k}\,\frac{\hbar}{i}\,I \tag{1.2}$$

Annihilation and creation operators:

$$a := \frac{\sqrt{2}}{2}\left(Q + \frac{i}{\hbar}P\right) \;\Rightarrow\; a^\dagger = \frac{\sqrt{2}}{2}\left(Q - \frac{i}{\hbar}P\right) \tag{1.3}$$

One computes:

$$[a_j, a_k^\dagger] = \delta_{jk}\ I. \tag{1.4}$$

The number operator and harmonic oscillator hamiltonian are:

$$N := \sum_{j=1}^{n} a_j^\dagger a_j\ , \qquad \hat{H} := \hbar\Big(N + \frac{1}{2}\Big) = \frac{\hbar}{2}\sum_{j=1}^{n}\Big(a_j a_j^\dagger + a_j^\dagger a_j\Big). \tag{1.5}$$

If we want to bring in the frequency of the oscillator we proceed as follows. In case $n = 1$:

$$\hat{H} := \omega\hbar\Big(N + \frac{1}{2}\Big) \qquad \text{where} \quad \omega = \frac{2\pi}{T}, \quad T \text{ the period.} \tag{1.6}$$

The oscillator in n degrees of freedom can be interpreted as a system of n one-dimensional oscillators. If the frequency of the j-th oscillator is ω_j, the total quantum Hamiltonian is

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\Big(a_j a_j^\dagger + a_j^\dagger a_j\Big). \tag{1.7}$$

The case when $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$ is called *the resonant case* and then

$$\hat{H} = \omega\hbar\Big(N + \frac{1}{2}\Big). \tag{1.8}$$

Lemma 1.1. *In n degrees of freedom, the spectrum of N consists entirely of eigenvalues: There exists an orthonormal basis of the Hilbert space, $\mathcal{H}$, of the form*

$$|\overline{k}{>}, \quad \overline{k} = (k_1, \cdots, k_n) \ \textit{a multi-index with } k_j \geq 0.$$

These are eigenvectors of N:

$$N|\overline{k}{>} = \Big(\sum_{j=1}^{n} k_j\Big)|\overline{k}{>}.$$

These are also eigenvectors of a system of n *uncoupled harmonic oscillators; the eigenvalue equation is:*

$$\hat{H}|\overline{k}> = \hbar\Big(\sum_{j=1}^{n} \omega_j\,(k_j + 1/2)\Big)\;|\overline{k}>.$$

Moreover, these eigenfunctions are obtained from the ground state, $|0>$*, by:*

$$|\overline{k}> = \frac{1}{\sqrt{\overline{k}!}}\,\Big((a_1^\dagger)^{k_1}\circ\cdots\circ(a_n^\dagger)^{k_n}\Big)|0>, \tag{1.9}$$

where $\overline{k}! = k_1!\cdots k_n!$.

For an isotropic system ($\omega_j = \omega$ for all j) the ground state, in $L^2(\mathbb{R}^n)$, is:

$$|0> = (\pi)^{-n/4}\,e^{-|x|^2/2}. \tag{1.10}$$

1.3. **Bargmann Space.** The definition:

$$\mathcal{B} = \{\ f:\mathbb{C}^n\to\mathbb{C}\ ;\ f \text{ entire and } \int |f(z)|^2\,e^{-|z|^2/\hbar}\,dm_z < \infty\ \}. \tag{1.11}$$

Here

$$dm_z = dxdy, \qquad z = x+iy \quad \text{vector notation.}$$

The inner product on $\mathcal{B}$ is:

$$\langle f,g\rangle = \int f(z)\,\overline{g(z)}\,e^{-|z|^2/\hbar}\,dm_z. \tag{1.12}$$

Lemma 1.2. *The adjoint of* $\frac{\partial}{\partial z_j}$ *is:*

$$\Big(\frac{\partial}{\partial z_j}\Big)^\dagger = \frac{1}{\hbar}\ \textit{multiplication by } z_j.$$

In Bargmann space the creation and annihilation operators can be taken to be:

$$a_j := \sqrt{\hbar}\,\frac{\partial}{\partial z_j}, \quad\Rightarrow\quad a_j^\dagger = \frac{1}{\sqrt{\hbar}}\,z_j. \tag{1.13}$$

Notice that indeed $[a_j, a_k^\dagger] = \delta_{jk}I$. The ground state is clearly a constant function. A calculation of the normalization shows that:

$$|0> = \Big(\frac{\hbar}{\pi}\Big)^{n/2}. \tag{1.14}$$

Therefore, by (1.9), the eigenfunctions of the number operator are simply monomials:

$$|\overline{k}> = \Big(\frac{\hbar}{\pi}\Big)^{n/2}\,\frac{1}{\sqrt{\overline{k}!}}\,\frac{1}{\hbar^{|\overline{k}|/2}}\,z_1^{k_1}\cdots z_n^{k_n}$$

Notice, by the way, that in Bargmann by (1.13) it is consistent to define:

$$Q = \frac{\sqrt{2}}{2}\left(\partial_z + z\right), \; P = \frac{\sqrt{2}}{2i}\Big(\partial_z - z\Big).$$

1.4. **The Bargmann Transform.**

2. The Heisenberg group

We've seen that the operators

$$Q_j = \text{mult. by } x_j, \qquad P_j = \frac{\hbar}{i}\,\frac{\partial}{\partial x_j}$$

satisfy the canonical commutation relations (CCR):

$$[P_j\,,\,Q_k] = \delta_{jk}\,\frac{\hbar}{i}, \quad \text{all other commutators zero.}$$

P_j is the observable "j-th component of the momentum" and Q_j is the observable "j-th component of the position".

Definition 2.1. Let (V,ω) a symplectic vector space of dimension $2n$. Then, on $V\times\mathbb{R}$, define de Lie algebra bracket

$$[(v,s)\,,\,(w,t)] := (0,\omega(v,w)).$$

The resulting Lie algebra is the Heisenberg Lie algebra of V.

This Lie algebra is *represented* by the operators Q,P above. Specifically, let $(e_1,\dots e_n,f_1,\dots f_n)$ be a canonical basis of (V,ω), that is, a basis such that

$$\omega(f_j,e_k) = \delta_{jk},$$

that is, if $(q_1,,\dots q_n,p_1,\dots p_n)$ are the associated (linear) coordinates

$$\omega = \sum dp_j\wedge dq_j.$$

Definition 2.2. The infinitesimal Schrödinger representation of the Heis. Lie algebra is defined as:

$$d\rho_h(e_j,0) = iQ_j, \quad d\rho_h(f_j,0) = iP_j = \hbar\frac{\partial}{\partial x_j}, \quad d\rho_\hbar(0,1) = i\hbar I.$$

Remark 2.3. The operators $d\rho_h$ are *skew-Hermitian.*

Back to the Heisenberg group:

Lemma 2.4. *The associated Lie group (the Heisenberg group) is isomorphic to $V \times \mathbb{R}$ with the group law*

$$(v, s) \cdot (w, t) = (v + w, s + t + \frac{1}{2}\omega(v, w)).$$

Proof. First of all, the Heis. Lie algebra is isomorphic to the Lie algebra of matrices of the form

$$m(p, q, s) := \begin{pmatrix} 0 & p & t \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}$$

(notation for $n = 1$). A simple calculation shows that the exponential of this matrix is:

$$\exp(m(p, q, t)) = I + m(p, q, t) + \frac{1}{2}m(0, 0, pq),$$

that is:

$$\exp(m(p, q, t)) = \begin{pmatrix} 1 & p & t + pq/2 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$$

A further calculation shows that

$$\exp(m(p, q, t)) \cdot \exp(m(p', q', t)) = \exp(m(p + p', q + q', t + t' + (pq' - qp')/2)$$

□

Next we integrate the Schrödigner representation of the Heisenberg Lie algebra. The resulting representation of the Heis. Lie group will be denoted: $\rho_\hbar$ Recall what it means to integrate a representation of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra with Lie group G, and exponential map $\exp : \mathfrak{g} \to G$. Let $d\rho$ be a rep. of $\mathfrak{g}$. Then the integrated representation is defined by the property that

$$\forall X \in \mathfrak{g} \qquad \frac{d}{dt}\rho(\exp(tX))(f) = d\rho(X)[\rho(\exp(tX))(f)]$$

for all vectors f in the representation space of $d\rho$. This is an equation on the curve $g(t) := \rho(\exp(tX))(f)$, and to find $\rho(\exp(X))$ given $d\rho$ one must solve it and set $t = 1$.

Back to the Schrodinger representation, first notice that $\{(0, 0, *), * \in \mathbb{R}\}$ is the center of the H. group.

Lemma 2.5. *Integrating the 1-parameter subgroup through $(0, 1) \in V \times \mathbb{R}$ gives:*

$$\rho_h(0, t)(f)(x) = e^{it\hbar}f(x).$$

Proof. The exponential of the Lie algebra element $t(0,0,1)$ is just $(0,0,t)$ so the equation that $g(t,x) := \rho_h(0,t)(f)(x)$ satisfies is:

$$\frac{\partial g}{\partial t} = d\rho_h(0,1)(g) = i\hbar g(t,x).$$

□

Proposition 2.6. *For all $f \in L^2(\mathbb{R}^n)$, the operators*

$$\rho_\hbar(p,q,u)(f)(x) := e^{iu\hbar + iq\cdot x + i\hbar p\cdot q/2} f(x+\hbar p)$$

form a Unitary representation of the Heisenberg group, it's the exponential of the Schrödinger representation of the H. Lie algebra.

Proof. To prove the formula above it is enough to consider the case: $u = 0$ (why?). Fix then (q,p) and consider the equation defining

$$g(x,t) := \rho_h(\exp(tq,tp,0))(f)(x),$$

namely:

$$\frac{\partial g}{\partial t} - \hbar \sum p_j \frac{\partial g}{\partial x_j} = i \sum_j q_j x_j g.$$

We must solve this equation. The left-hand side is the directional derivative of $g(x,t)$ with respect to the vector $(-hp,1)$. This suggests to change coordinates from (x,t) to (y,s) where $y = x + t\hbar p$ and $s = t$ (method of characteristics). Then, since $\partial t/\partial s = 1$ and $\partial x/\partial s = -\hbar p$:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial t} - \hbar p \cdot \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x}$$

Therefore, if G denotes g in the new coordinates, the equation that G satisfies is

$$\frac{\partial G}{\partial s} = iq\cdot(y - s\hbar p)\, G.$$

This immediately integrates to:

$$G(s,y) = e^{isq\cdot y - is^2\hbar q\cdot p/2}\, G(0,y).$$

Going back to the (x,t) coordinates and setting $t = 1$ yields the result. (Try it!). □

3. The Metaplectic Representation

3.1. **The symplectic Lie group and Lie algebra.** Take $V = \mathbb{R}^{2n}$ with its standard symplectic form:

$$\omega(u,v) = u^t \cdot J \cdot v$$

where u, v are thought to be column vectors, multiplication is matrix multiplication and J has the $n \times n$ block decomposition

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

If (q,p) denote the standard coordinates on V then $\omega = \sum dp \wedge dq$.

Definition 3.1. The symplectic group, $\mathrm{Sp}(n)$, is the group of linear transformations $T : V \to V$ such that $\forall u,\ v \in V\ \omega(Tu, Tv) = \omega(u,v)$. The symplectic Lie algebra, $\mathrm{sp}(n)$, is the space of all linear transformations $A : V \to V$ such that, $\forall t \in \mathbb{R},\quad \exp(tA) \in \mathrm{Sp}(n)$.

Lemma 3.2. *A linear transformation $A : V \to V$ is in sp iff*

$$\forall u, v \in V \qquad \omega(Au, v) + \omega(u, Av) = 0.$$

If we identify A with an $n \times n$ matrix, this is equivalent to: $A^tJ + JA = 0$. If we write A in $n \times n$ blocks,

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then

$$A \in sp \quad \Leftrightarrow \quad \gamma,\ \beta \text{ are symmetric and } \alpha = -\delta^t.$$

In particular the dimension of $sp(n)$ is $n(2n+1)$.

Lemma 3.3. *Let S^2V denote the space of quadratic forms on V. Then S^2V is a subalgebra of the Lie algebra $(C^\infty(V), \{,\})$. Moreover, the map*

$$\begin{array}{ccl} sp(n) & \to & S^2V \\ A & \mapsto & Q_A(v) = \frac{1}{2}\omega(Av, v) \end{array}$$

is a Lie algebra anti-isomorphism (i.e. $\{Q_A, Q_B\} = -Q_{[A,B]}$).

Remark 3.4. Let $A \in \mathrm{sp}(n)$. Clearly, the differential of Q_A at v is

$$d(Q_A)_v(w) = \frac{1}{2}\omega(Av, w) + \frac{1}{2}\omega(Aw, v) = \omega(Av, w).$$

Therefore, the Hamilton vector field of Q_A is

$$\left(\Xi_{Q_A}\right)_v = -Av.$$

We will now discuss a particularly important subalgebra of sp: the algebra of the unitary group. Notice that the matrix J above has the property: $J^2 = -I$. In fact the $\mathbb{R}$-linear isomorphism

$$\mathbb{R}^{2n} \ni (q,p) \mapsto z = q + ip \in \mathbb{C}^n \tag{3.1}$$

intertiwnes J and "multiplication by i". We can use this isomorphism to identify complex-linear transformations of $\mathbb{C}^n$ with certain $\mathbb{R}$-linear transformations of $\mathbb{R}^{2n}$, namely those that commute with J:

$$\mathrm{GL}(n,\mathbb{C}) = \{\, T : V \to V \mid T \circ J = J \circ T \,\}. \tag{3.2}$$

Not all these transformations are symplectic. The situation is as follows:

Lemma 3.5. *$GL(n,\mathbb{C}) \cap Sp(n) = U(n)$, the unitary group. One also has:*

$$O(2n) \cap Sp(n) = U(n) = O(n) \cap GL(n,\mathbb{C}).$$

Lemma 3.6. *Let $u(n)$ be the Lie algebra of $U(n)$ (so $A \in u(n)$ iff $\forall t \ \ \exp(tA) \in U(n)$. Then, $\forall A \in sp(n)$, $A \in u(n) \Leftrightarrow A$ is skew-symmetric $\Leftrightarrow Q_A(z) = \overline{z}^t\, M\, z$, where M is a Hermitian $n \times n$ matrix.*

Exercise 3.1.1. Show that the matrices in $u(n)$, when viewed as real $2n \times 2n$ matrices via (3.1), are precisely of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with α skew-symmetric and β symmetric. Show that the $n \times n$ complex matrix associated to the above is $\alpha + i\beta$. Conclude that the polynomials

$$\begin{cases} q_j q_k + p_k p_j & 1 \le j \le k \le n, \\ q_j p_k - q_k p_j & 1 \le j < k \le n \end{cases} \tag{3.3}$$

form a basis of the subspace of S^2V corresponding to $u(n)$.

Given that $u(n)$ are the skew-symmetric elements of $u(n)$, it makes sense to introduce:

$$\mathfrak{p} := \{\, A \in \mathrm{sp}(n) \mid A^t = A \,\}, \tag{3.4}$$

the symmetric matrices in sp. Clearly

$$\mathrm{sp}(n) = u(n) \oplus \mathfrak{p}.$$

Lemma 3.7. 1. $[u(n), u(n)] \subset u(n)$, $\quad [u(n), \mathfrak{p}] \subset \mathfrak{p}$, $\quad [\mathfrak{p}, \mathfrak{p}] \subset u(n)$.

2. *$A \in \mathfrak{p} \Leftrightarrow$ the real quadratic form Q_A is the real (or imaginary) part of a complex quadratic form on $\mathbb{C}^n$.*

Example 3.8. If $n = 1$ then the quadratic forms corresponding to $u(1)$ are all multiples of $p^2 + q^2$, while $\mathfrak{p}$ is two-dimensional, spanned by $p^2 - q^2$ and qp.

3.2. **The infinitesimal representation.** It turns out that one can extend the Heisenberg quantization rules,

$$q \mapsto Q, \quad p \mapsto P$$

to quadratic Hamiltonians and obtain a representation of sp.

Lemma 3.9. *The linear map, τ, from S^2V to operators on $\mathcal{H}$, defined by:*

$$\begin{array}{lll} q_j q_k & \mapsto & iQ_jQ_k \\ p_j p_k & \mapsto & iP_jP_k \\ q_j p_k & \mapsto & \frac{i}{2}\Big(Q_jP_k + P_kQ_j\Big), \end{array} \tag{3.5}$$

is an anti-representation of $(S^2V, \{\,\})$. Therefore, $d\mu$ defined by:

$$sp(n) \ni A \mapsto d\mu(A) := \tau(Q_A)$$

is a representation of $sp(n)$.

Notice that the operators (3.5) are indeed formally skew-adjoint. More importantly, notice that the definition is consistent (with respect to: composition of operators not being commutative in general).

Next we turn to the problem: does $d\mu$ exponentiate to Sp?

3.3. **The representation of $U(n)$.** We will now show that the restriction of $d\mu$ to $U(n)$ exponentiates to the *double cover* of $U(n)$. Let's first write the quantization of the Hamiltonians (3.3). The first line in (3.3) quantizes to the operator

$$X_{jk} := i\left(Q_jQ_k + P_jP_k\right), \tag{3.6}$$

while the second quantizes to

$$Y_{jk} := \frac{i}{2}\left(Q_jP_k + P_kQ_j\right) - \frac{i}{2}\left(Q_kP_j + P_jQ_k\right). \tag{3.7}$$

At this point it is more convenient to work in Bargmann space. The question we consider first is: what is the infinitesimal representation $d\mu$ of $u(n)$ in Bargmann space? For simplicity I will take $\hbar = 1$.

Consider the creation and annihilation operators, a, $a^\dagger$. From their definition, (1.3), we see that.

$$Q = \frac{\sqrt{2}}{2}\left(a + a^\dagger\right), \quad P = \frac{\sqrt{2}}{2i}\left(a - a^\dagger\right)$$

(omitting indices). A calculation shows:

$$X_{jk} = i\left(a_ja_k^\dagger + a_j^\dagger a_k\right), \tag{3.8}$$

while

$$Y_{jk} = \frac{1}{2}\left(a_j^\dagger a_k - a_j a_k^\dagger + a_k a_j^\dagger - a_k^\dagger a_j\right). \tag{3.9}$$

More explicitly, with an obvious notation:

$$\frac{1}{i} X_{jk} = \partial_j z_k + z_j \partial_k = z_k \partial_j + z_j \partial_k + \delta_{jk}, \tag{3.10}$$

$$Y_{jk} = z_j \partial_k - z_k \partial_j. \tag{3.11}$$

Notice that *these are first-order operators.* This means that the exponential of the restriction of $d\mu$ to $u(n)$ is given by a geometrical action on $\mathbb{C}^n$:

Theorem 3.10. *Let $\tilde{U}$ the double cover of $U(n)$. The rule: $\forall g \in \tilde{U},\ f \in \mathcal{B}$*

$$\mu(g)(f)(z) = \frac{1}{\det g^{1/2}} f(g^{-1} \cdot z)$$

defines a representation of the double cover of $U(n)$ whose infinitesimal representation is given by (3.10,3.11). Here g acts on $\mathbb{C}^n$ via its image in $U(n)$.

Example 3.11. Before turning to the proof of the Theorem, it is instructive to consider the important example of the harmonic oscillator. The classical Hamiltonian is:

$$H = \frac{1}{2}\sum_{j=1}^{n}\left(q_j^2 + p_j^2\right) = \frac{1}{2}|z|^2.$$

The matrix in $\mathrm{sp}(n)$ corresponding to H is J itself. It is easy to check that, in complex notation, the one-parameter group, (g_t), of linear symplectic transformations generated by H is:

$$g_t \cdot z = e^{-it} z.$$

Therefore, according to Theorem 3.10, the representation of (g_t) is the one-parameter family of operators

$$\mu(g_t)(f)(z) = e^{int/2}\, f(e^{it} z).$$

Differentiating this and setting $t = 0$ we get:

$$\frac{in}{2} f(z) + i \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} \tag{3.12}$$

On the other hand the operator associated to J by $d\mu$ is

$$d\mu(J) = \frac{i}{2}\sum_{j=1}^{n}\left(Q_j^2 + P_j^2\right) = \frac{i}{2}\sum_{j=1}^{n} X_{jj} = \frac{i}{2}\sum_{j=1}^{n}\left(2 z_j \partial_j + 1\right),$$

the last equality by (3.10). We see that indeed

$$\frac{d}{dt}\Big(e^{int/2}\ f(e^{it}z)\Big)|_{t=0} = d\mu(J)(f).$$

3.4. **The representation of $Mp(n)$.**

4. Weyl Quantization

4.1. **The Fourier transform.** We briefly recall here some elementary facts about the Fourier transform. We begin with the Fourier inversion formula: If f is a function on $\mathbb{R}^N$

$$f(u) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{f}(v)\, e^{iv\cdot u}\, du, \tag{4.1}$$

where

$$\hat{f}(v) = \int_{\mathbb{R}^N} \hat{f}(v)\, e^{-iv\cdot w}\, dw$$

is the Fourier transform. This formula is valid for the Schwartz class of functions, f, that can be described in terms of the Heisenberg representation as follows.

Definition 4.1. Let $\rho : G \to U(\mathcal{H})$ is a unitary representation of a Lie group G. Then a vector $f \in \mathcal{H}$ is called a C^∞ vector iff $\forall \phi \in \mathcal{H}$ the function on G

$$G \ni g \mapsto \langle \rho(g)(f), \phi \rangle$$

is of class C^∞. If G is the Heisenberg group and $\mathcal{H} = L^2(\mathbb{R}^n)$, the space

$$\mathcal{S}(\mathbb{R}^n) := \{\, f \in L^2(\mathbb{R}^n)\ ;\ f \text{ is a smooth vector}\,\}$$

is the Schwartz space.

The following lemma characterizes $\mathcal{S}$, and usually serves as the definition of $\mathcal{S}$.

Lemma 4.2. *A function f on $\mathbb{R}^n$ is in $\mathcal{S}$ iff it is smooth and $\forall \alpha, \beta$ multi-indices $\exists C_{\alpha,\beta} > 0$ such that*

$$\forall x \in \mathbb{R}^n \quad |x^\alpha D^\beta f(x)| < C_{\alpha,\beta}.$$

One introduces on $\mathcal{S}$ the family of semi-norms

$$|||f|||_M \;=\; \sum_{|\alpha \leq M,\ |\beta| \leq M} \ \sup_{x\in\mathbb{R}^n} |\, x^\alpha\, D^\beta f(x)\,|\,.$$

With these semi-norms $\mathcal{S}$ becomes a *topological vector space.*

Definition 4.3. A tempered distribution is a linear functional

$$\begin{array}{rcl} T : \mathcal{S} & \to & \mathbb{C} \\ f & \to & < T, f > \end{array}$$

such that $\exists M,\ C > 0$ such that

$$\forall f \in \mathcal{S} \quad |< T, f >| \leq C\, |||f|||_M. \tag{4.2}$$

The set of all tempered distributions will be denoted: $\mathcal{S}'$.

We recall also the following property of the Fourier transform:

Lemma 4.4. *$\forall f,\, g \in C_0^\infty$ and all multi-indices α,*

1. $\widehat{D^\alpha f}(\xi) = \xi^\alpha\, \hat{f}(\xi)$,
2. $\widehat{x_j f}(\xi) = i\dfrac{\partial}{\partial \xi_j}\hat{f}$,
3. $\displaystyle\int \hat{f}(y)\, g(y)\, dy = \int f(x)\, \hat{g}(x)\, dx$.

4.2. **Definition of Weyl quantization.** Recall that the Heisenberg representation gives a meaning to the exponential of the operator

$$i\Big(q\cdot Q + p\cdot P\Big) = i\sum_{j=1}^n \, q_j x_j + p_j \frac{\hbar}{i}\,\frac{\partial}{\partial x_j},$$

namely:

$$e^{i(q\cdot Q+p\cdot P)}(f)(x) = e^{iq\cdot x + i\hbar p\cdot q/2}\, f(x+\hbar p), \tag{4.3}$$

see Proposition (2.6). Therefore, in analogy with (4.1), one can make the following formal definition:

Definition 4.5. Let $a(q,p)$ be a function on $\mathbb{R}^{2n}$. The **Weyl quantization** of a is the operator defined by:

$$\tilde{a}^w = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \hat{a}(q,p)\, e^{i(q\cdot Q+p\cdot P)}\, dq\, dp$$

where the exponential is the operator (4.3).

The convergence of the integral above is in the *weak sense*, meaning the following: For every $f\in C_0^\infty(\mathbb{R}^n)$, by definition the action of $\tilde{a}^w$ on f is the function

$$\tilde{a}^w(f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \hat{a}(q,p)\, e^{iq\cdot x+i\hbar p\cdot q/2}\, f(x+\hbar p)\, dp\, dq. \tag{4.4}$$

This integral converges if, for example, f is rapidly decreasing and a is rapidly decreasing in q. But we can extend the definition to a's of tempered growth, by duality, as the following example shows.

Example 4.6. Let $a(q,p) = p_j$. Its Fourier transform is defined by duality: For all $u\in\mathcal{S}(\mathbb{R}^{2n})$

$$\int \hat{a}(z)\, u(z)\, dz := \int a(w)\, \hat{u}(w)\, dw = \int p_j\, \hat{u}(q,p)\, dq\, dp = -i(2\pi)^n\,\frac{\partial u}{\partial p_j}(0,0),$$

where the last equality follows from the Fourier inversion formula. Applying this identity to (4.4) one sees that

$$\tilde{p_j}^w = P_j.$$

Similarly one can show that $\tilde{q_j}^w = Q_j$.

The following provides an alternative expression for the operator $\tilde{a}^w$ that will be useful in proving the product formula.

Lemma 4.7.

$$\tilde{a}^w(f)(x) = \frac{1}{(2\pi\,\hbar)^n}\int e^{i\hbar^{-1}(x-y)\cdot p}\, a\left(\frac{x+y}{2}, p\right) f(y)\, dp\, dy.$$

We now list some fundamental properties of the Weyl quantization.

Lemma 4.8. *If a is real-valued, $\tilde{a}^w$ is formally self-adjoint.*

Proof. This follows directly from the definition and the fact that a is real-valued iff $\hat{a}(-q,-p) = \overline{\hat{a}(q,p)}$. ☐

Lemma 4.9. *Weyl quantization is covariant with respect to the metaplectic representation, in the following sense. Let $g \in Sp(n)$, and let $a^g(z) = a(g^{-1}\cdot z)$. Then*

$$\left(\tilde{a^g}\right)^w = \mu(g)\circ\tilde{a}^w\circ\mu(g)^{-1}.$$

4.3. **The product formula.** Let us turn next to the computation of the composition of two operators, $\tilde{a}^w\circ\tilde{b}^w$. The result is as follows:

Proposition 4.10. *The composition of two Weyl operators is an operator of the form:*

$$(\tilde{a}^w\circ\tilde{b}^w)(f)(x) = \frac{1}{(2\pi\,\hbar)^n}\int e^{i\hbar^{-1}(x-y)\cdot p}\, c\left(\frac{x+y}{2}, p, \hbar\right) f(y)\, dp\, dy. \tag{4.5}$$

where, as $\hbar\to 0$

$$c(q,p,\hbar) \sim \exp\left(\frac{i\hbar}{2}(D_q\cdot D_\eta - D_y\cdot D_p)\right)\Big(a(q,p)b(y,\eta)\Big)|_{y=q;\ \eta=p}.$$

$$\sim a(q,p)b(q,p) + \frac{\hbar}{2i}\{a\,,\,b\}(q,p) + \cdots$$

Notice that by this result, we are led to consider $\hbar$-dependent "symbols", $a(q,p,\hbar)$, that have an asymptotic expansion in increasing powers of $\hbar$:

$$a(q,p,\hbar) \sim \sum_{j=0}^{\infty}\hbar^j\, a_j(q,p). \tag{4.6}$$

Before proving the product formula, we note that it has the following algebraic consequence. Define, on the algebra of formal power series in $\hbar$,

$$\mathcal{A} := C^\infty(\mathbb{R}^{2n})[[\hbar]]$$

the following product: $\forall\, \alpha,\ \beta \in \mathcal{A}$

$$\alpha * \beta = \exp\left(\frac{i\hbar}{2}(D_q \cdot D_\eta - D_y \cdot D_p)\right)\Big(\alpha(q,p)\beta(y,\eta)\Big)|_{y=q;\ \eta=p}$$

where the exponential is to be interpreted as a formal power series of differential operators acting on the C^∞ coefficients of α, β. Then one has:

Proposition 4.11. *The $*$ product defined above is associative, and furthermore $\forall \alpha,\beta \in C^\infty$ $\alpha * \beta - \beta * \alpha = \frac{h}{i}\,\{a\,,\,b\} + O(\hbar^2)$ where $\alpha = a + O(\hbar)$, $\beta = b + O(\hbar)$.*

This algebraic structure **is** *the most general manifestation of the existence of quantization of observables.*

We now indicate how one proves Proposition 4.10. By lemma 4.7, the Schwartz kernel of $\tilde{a}^w \circ \tilde{b}^w$ is:

$$K(x,y) = \frac{1}{(2\pi\,\hbar)^{2n}} \int e^{i\hbar^{-1}(x-w)\cdot\xi}\, e^{i\hbar^{-1}(w-y)\cdot\eta}\, a\left(\frac{x+w}{2},\xi\right) b\left(\frac{w+y}{2},\eta\right)\, d\xi\, d\eta\, dw.$$

Make the change of variables:

$$u := x - y, \quad v := x + y$$

and define F to be the expression of K in the new variables:

$$F(u,v) = K\left(\frac{u+v}{2}\,,\,\frac{v-u}{2}\right)$$

For the composition theorem, we want to write F in the form:

$$F(u,v) = \frac{1}{(2\pi\hbar)^n} \int e^{\hbar^{-1}u\cdot z}\, \Phi(z,v,\hbar)\, dz.$$

(Indeed the composition theorem asserts that Φ admits a certain asymptotic expansion.) By a simple change of variables

$$F(u,v) = \frac{1}{(2\pi)^n} \int e^{iu\cdot z}\, \Phi(\hbar z, v, \hbar)\, dz,$$

and by the Fourier inversion formula we must have

$$\Phi(\hbar z, v, \hbar) = \int e^{-iz\cdot\zeta}\, F(\zeta, v)\, d\zeta,$$

that is

$$\Phi(z,v,\hbar) = \int e^{-i\hbar^{-1}z\cdot u}\, F(u,v)\, du.$$

Substituting F by its expression, we obtain

$$\Phi(z,v,\hbar) = \int e^{i\hbar^{-1}\phi(u,w,\xi,\eta)}\, a\left(\frac{u+v}{4}+\frac{w}{2},\xi\right)\, b\left(\frac{v-u}{4}+\frac{w}{2},\eta\right)\, d\xi\, d\eta\, dw\, du.$$

where

$$\phi(u,w,\xi,\eta) = \xi\cdot(u/2-w)+\eta\cdot(w+u/2)-z\cdot u.$$

Finally, one needs to apply the *method of stationary phase* to this integral.

5. The method of stationary phase

The method of stationary phase is the name given to a collection of old results on the asymptotic behavior of integrals of the form

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau\phi(x)}\, a(x)\, dx \tag{5.1}$$

where ϕ is smooth and for simplicity we take $a\in C_o^\infty(\mathbb{R}^n)$ and independent of τ. ϕ is called the phase and a the amplitude. We will choose ϕ to be real-valued. The problem is to estimate (5.1) as $\tau\to\infty$. Notice the obvious bound

$$|I(\tau)| \le \|a\|_{L^1}, \tag{5.2}$$

but $I(\tau)$ may well be smaller. As a warm-up let's look at two extreme cases.

1. Suppose ϕ is constant: $\phi(x)=c$. Then

$$I(\tau) = e^{i\tau c}A,$$

where A is a constant (i.e. independent of τ). Therefore $I(\tau)$ oscillates but $|I(\tau)|$ is a constant. This shows that the obvious bound (5.2) is optimal in general.

2. Suppose ϕ is a coordinate function, say $\phi(x)=x_1$. Then

$$I(\tau) = \hat{a}(-\tau,0,\dots,0).$$

Since a is smooth compactly supported $\hat{a}$ is Schwartz, and therefore in this case

$$\forall N>0 \qquad I(\tau) = O(\tau^{-N}).$$

Intuitively, in case 2 we obtain rapid decrease because for large τ the exponential $\exp(i\tau\phi(x))$ oscillates many times between x and $x+dx$, while the amplitude $a(x)$ remains relatively constant (since a is smooth). Therefore in the sum (integral) many cancellations take place. This is clearly the opposite of what happens in case 1. If ϕ has a critical point x_0, i.e. if $d\phi_{x_0}=0$, then between x_0 and x_0+dx the phase ϕ doesn't change very much, so infinitesimally we are in case 1. Therefore we expect that in general the main contributions to $I(\tau)$ should arise from the critical points of ϕ, that is points where ϕ is "stationary". This is indeed the case.

5.1. **Non-stationary phases.** First we show that the main contribution arise from the critical points of the phase in the support of the amplitude.

Proposition 5.1. *Let* $I(\tau) = \int_{\mathbb{R}^n} e^{i\tau\phi(x)}\, a(x)\, dx$ *where* $a \in C_o^\infty(\mathbb{R}^n)$ *and* ϕ *is smooth. Assume that* ϕ *doesn't have any critical points in a neighborhood of the support of* a*. Then*

$$\forall N>0 \qquad I(\tau) = O(\tau^{-N}).$$

Proof. The idea is to use the identity:

$$\nabla\phi\left(e^{i\tau\phi(x)}\right) = i\tau\|\nabla\phi\|^2 e^{i\tau\phi(x)}$$

and to integrate by parts. Here on the left-hand side $\nabla\phi$ is considered as the differential operator

$$\nabla\phi = \sum_{j=1}^n \frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}.$$

Let χ be a smooth cut-off function identically one on the support of a and such that $\nabla\phi$ does not vanish on its support. Therefore $a\chi = a$, and $\chi/\|\nabla\phi\|^2$ is a smooth function. Let

$$L := \frac{\chi}{\|\nabla\phi\|^2}\sum_{j=1}^n \frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}.$$

This is a smooth vector field satisfying

$$L\, e^{i\tau\phi(x)} = i\tau\,\chi\, e^{i\tau\phi(x)},$$

and therefore

$$a\, e^{i\tau\phi(x)} = a\,\chi\, e^{i\tau\phi(x)} = \frac{a}{i\tau}\, L\, e^{i\tau\phi(x)}.$$

Therefore

$$I(\tau) = \int \frac{a}{i\tau}\, L\, e^{i\tau\phi(x)}\, dx = \frac{i}{\tau}\int e^{i\tau\phi(x)}\, L^t a\, dx. \tag{5.3}$$

Here L^t is the transpose of L. Applying the bound (5.2), we get:

$$|I(\tau)| \leq \frac{1}{|\tau|} \, \|L^t a\|_{L^1}$$

Notice however that the right-hand side of (5.3) is of the same form as $I(\tau)$ itself (except for the factor of i/τ). Therefore the argument can be repeated any number of times. □

In applications it will be important to let both the phase and the amplitude depend on parameters, that is consider

$$I(\tau, w) \;=\; \int e^{i\tau\phi(x,w)} \, a(x,w) \, dx \tag{5.4}$$

where w ranges over some compact domain Ω in $\mathbb{R}^m$. Of course we can repeat e.g. the analysis above for each fixed w; the only question is that of the uniformity of the estimates with respect to ω.

Proposition 5.2. *Assume a and ϕ are smooth. Let $K \subset \mathbb{R}^n$ be a fixed compact set such that*

$$\forall w \in \Omega, \; \forall x \notin K \qquad a(x,w) \;=\; 0$$

(i.e. for each w the support of $a(\cdot, w)$ is contained in K). Assume that

$$\{(x,w) \in K \times \Omega \,;\, \nabla_x \phi(x,w) = \vec{0}\} \;=\; \emptyset.$$

Then

$$\forall N \qquad |I(\tau,w)| = O(\tau^{-N})$$

uniformly on $w \in \Omega$.

5.2. **Quadratic phases.** The simplest phases with a non-degenerate critical point are the quadratic ones, $\phi(x) \;=\; x\cdot Ax/2$ where A is a real non-degenerate symmetric matrix. For such phases one has the following result:

Theorem 5.3. *If a is smooth and compactly supported, then there exists a complete asymptotic expansion*

$$\int_{\mathbb{R}^n} e^{i\tau x\cdot Ax/2} \, a(x) \, dx \;\sim\; \left(\frac{2\pi}{\tau}\right)^{n/2} \frac{e^{i\pi\sigma/4}}{\sqrt{|\det A|}} \sum_{j=0}^{\infty} \alpha_j(a)(0) \, \tau^{-j} \tag{5.5}$$

as $\tau \to \infty$, where:

1. *σ is the signature of A (the number of positive eigenvalues minus the number of negative ones).*
2. *$\alpha_0(a)(0) = a(0)$, and in general*
3. *$\alpha_j(a)$ is a differential operator of order $2j$ applied to a.*

We need a preliminary result:

Proposition 5.4. *Let A be a symmetric positive-definite $n \times n$ matrix, and let*

$$g_A(x) \;=\; e^{-x\cdot Ax/2}.$$

(Think of x as a column vector, so that $x \mapsto x \cdot Ax$ is the quadratic form defined by A.) Then the FT of g_A is:

$$\widehat{g_A}(\xi) \;=\; \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\, g_{A^{-1}}(\xi).$$

We turn now to the proof of the Theorem.

Proof. We can assume without loss of generality that A is diagonal, and then by a density argument (of product functions in $C_0^\infty(\mathbb{R}^n)$) it suffices to prove the result for $n = 1$. So take $A = (\Lambda)$, $\Lambda \in \mathbb{R}$. First assume that $\Lambda < 0$. By Parseval's identity and Proposition 5.4, $\forall \mu > 0$

$$\int e^{\mu x^2 \Lambda/2}\, a(x)\, dx \;=\; \frac{\sqrt{2\pi}}{2\pi\sqrt{-\mu\Lambda}} \int e^{y^2/2\Lambda\mu}\, \hat{a}(y)\, dy.$$

The advantage of doing this is that on the right-hand side the μ appears in the denominator. Now regard the integrals above as a function of μ. Clearly this function of μ is analytic, so one can analytically continue it to $\mu = i\tau$, $\tau > 0$, that is from the positive real semi-axis to the positive imaginary semi-axis. We simply have to take care of taking the correct analytic continuation of the square root $\sqrt{-\mu\Lambda} = \sqrt{\mu}\sqrt{|\Lambda|}$ (recall that we are assuming $\Lambda < 0$). This leads to the identity

$$\int e^{i\tau x^2\Lambda/2}\, a(x)\, dx \;=\; \frac{1}{\sqrt{2\pi}\, e^{i\pi/4}\sqrt{\tau}\,\sqrt{|\Lambda|}} \int e^{iy^2/2\Lambda\tau}\, \hat{a}(y)\, dy.$$

This is what we had to show, i.e. it's (5.5) for the case at hand. If, on the other hand, we take $A = (\Lambda)$ with $\Lambda > 0$ then we repeat the above argument starting from $\int e^{-\mu x^2\Lambda/2}\, a(x)\, dx$. Now we have to analytically continue from the positive real semi-axis to the negative imaginary semi-axis, so we obtain a factor of $\exp -i\pi/4$ in the denominator. In conclusion one arrives at the identity

$$\int_{\mathbb{R}^n} e^{i\tau x\cdot Ax/2}\, a(x)\, dx \;=\; \frac{e^{i\pi\sigma/4}}{(2\pi)^{n/2}\,\tau^{n/2}\,\sqrt{|\det A|}} \int e^{iy\cdot A^{-1}y/2\tau}\, \hat{a}(y)\, dy. \tag{5.6}$$

Now do a Taylor-McLaurin expansion of the exponential and use the Fourier inversion formula. $\square$

5.3. **Non-degenerate critical points.** We next consider $I(\tau)$ where the phase function, ϕ, only has non-degenerate critical points in a neighborhood of the support of the amplitude. Although it is possible to analyze $I(\tau)$ in more general cases, the non-degenerate case is by far the simplest and basically the only one we'll consider here.

We begin by recalling the definition of a non-degenerate critical point:

Definition 5.5. A point $p \in \mathbb{R}^n$ is a non-degenerate critical point of $\phi : \mathbb{R}^n \to \mathbb{R}$ iff $\nabla\phi(p) = \vec{0}$ and the Hessian of ϕ at p,

$$\mathrm{Hess}_p(\phi) \;:=\; \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(p)\right)$$

is non-degenerate: $\det(\mathrm{Hess}_p(\phi)) \neq 0$.

It is known that non-degenerate critical points must be isolated (among all critical points). This is in fact implied by the following "Morse lemma", which says that near a non-degenerate critical point a function is equivalent to a quadratic function:

Lemma 5.6. *Let p be a non-degenerate critical point of a smooth function ϕ. Then there exists a neighborhood of p, W, a neighborhood of the origin in $\mathbb{R}^n$, X, and a diffeomorphism $\chi : X \to W$ such that $\chi(0) = p$ and*

$$\phi(\chi(y_1, \dots, y_n)) \;=\; \phi(p) + \frac{1}{2}\sum_{j=1}^{n} \Lambda_j y_j^2$$

where the Λ_j are the eigenvalues of $\mathrm{Hess}_p(\phi)$. Moreover, the Jacobian of χ at the origin is one.

Combining this Lemma with Theorem 5.3 and Proposition 5.1, we obtain the following result:

Theorem 5.7. *Suppose that ϕ is a smooth, real-valued function on $\mathbb{R}^n$, and let $a \in C_0^\infty(\mathbb{R}^n)$. Suppose that the only critical points of ϕ in some neighborhood of the support of a are non-degenerate, and denote those in the support of a by $p_1, \dots, p_K$. Then, as $\tau \to \infty$*

$$\int e^{i\tau\phi(x)}\, a(x)\, dx \sim \left(\frac{2\pi}{\tau}\right)^{n/2} \sum_{l=1}^{K} \frac{e^{i\pi\sigma_l/4}}{\sqrt{|\det \mathit{Hess}_{p_l}(\phi)|}}\, e^{i\tau\phi(p_l)} \left[a(p_l) + \sum_{j=1}^{\infty} \alpha_j^l(a)(p_l)\, \tau^{-j}\right]$$

where σ_l is the signature of $\mathit{Hess}_{p_l}(\phi)$ and the α_j^l are differential operators.

The restriction that all critical points of the phase be non-degenerate is very restrictive. However, the following situation often occurs. Suppose the variables x

split: $x = (x', x'')$ and suppose that every point along the plane $x' = 0$ is a critical point of the phase $\phi(x)$: $\nabla_x \phi(0, x'') = 0$. Suppose moreover that the Hessian in the x' variables is non-degenerate at each $(0, x'')$:

$$\det\left(\frac{\partial^2 \phi}{\partial x'_i \partial x'_j}(0, x'')\right) \neq 0.$$

Clearly we can write

$$I(\tau) := \int e^{i\tau\phi(x)}\, a(x)\, dx = \int \left(\int e^{i\tau\phi(x)}\, a(x)\, dx'\right) dx''.$$

Then we can do stationary phase on the dx' integral, for each x'', and (formally at least) integrate the resulting expansion with respect to dx'' to estimate $I(\tau)$. This in fact can be made rigorous, but I won't write the details here.

6. Path Integrals

6.1. **A "derivation".** Finally:

$$< q''|e^{-iT\tilde{H}}|q'> = \int_{\mathcal{P}(T,q',q'')} e^{i\int_0^T [p\cdot\dot{q}-h(p,q)]dt}\ \Pi_t \frac{dq(t)dp(t)}{(2\pi)^n} \tag{6.1}$$

where $\mathcal{P}(T,q',q'')$ is the space of paths

$$\mathcal{P}(T,q',q'') = \{\ (p,q) : [0,T] \to \mathbb{R}^{2n}\ ;\ q(0) = q',\ q(T) = q''\ \}.$$

There are simplifications if the PQ hamiltonian $h(p,q)$ is quadratic in p. Suppose that indeed

$$h(p,q) = \frac{1}{2m}\|p\|^2 + V(q).$$

Then, completing the square, the phase of the integrand in (6.1) can be written as:

$$\int_0^T \left[-\frac{1}{2m}\|p - m\dot{q}\|^2 + \frac{1}{2m}\|\dot{q}\|^2 - V(q(t)) \right]\ dt.$$

We can then imagine doing the p-integrations first in (6.1), obtaining an expression of the form

$$< q''|e^{-iT\tilde{H}}|q'> = \int_{\mathcal{Q}(T,q',q'')} e^{i\int_0^T [\frac{1}{2m}\|\dot{q}\|^2 - V(q(t))]dt}\mathcal{D}q(t) \tag{6.2}$$

where

$$\mathcal{Q}(T,q',q'') = \{\ q : [0,T] \to \mathbb{R}^n\ ;\ q(0) = q',\ q(T) = q''\ \}.$$

Notice that the phase in (6.2) is precisely the classical action.

Exercise 6.1.1. By analogy with the previous derivation argue that

$$< q''|e^{-T\tilde{H}}|q'> = \int_{\mathcal{Q}(T,q',q'')} e^{-\int_0^T [\frac{1}{2m}\|\dot{q}\|^2 + V(q(t))]dt}\ \mathcal{D}q(t). \tag{6.3}$$

6.2. **An example.** The path integral is a powerful computational tool, although it is not rigorously defined. Almost miraculously, the calculations often provide exact, beautiful formulae. We will now present an example of such a calculation.

Let

$$\tilde{H} = \frac{1}{2}\left(P^2 + Q^2\right)$$

be the quantum harmonic oscillator with $n = 1$. Let's raise the problem of computing the trace $\mathrm{Tr}\, e^{-T\tilde{H}}$, with T a fixed number. Since we know the spectrum of

$\tilde{H}$, we can immediately write

$$\operatorname{Tr} e^{-T\tilde{H}} = \sum_{n=0}^{\infty} e^{-T(n+1/2)} = \frac{1}{2\sinh T/2}.$$

On the other hand, by (6.3), one can write

$$\operatorname{Tr} e^{-T\tilde{H}} = \int_{\mathcal{L}_T} e^{-\frac{1}{2}\int_0^T [\dot{q}^2+q^2]dt}\,\mathcal{D}q(t)$$

where $\mathcal{L}_T = \{\, q(t) : S^1_T \to \mathbb{R}\}$ where S^1_T is the circle of diameter T. We are going to regard this path integral as an infinte-dimensional analogue of a Gaussian integral. If Λ is an $N \times N$ symmetric matrix, then

$$\int_{\mathbb{R}^N} e^{-x\cdot\Lambda x/2}\,\frac{dx}{(2\pi)^{N/2}} = \frac{1}{\sqrt{\det\Lambda}}$$

To develop the analogy, we must first write the integral $\int_0^T \left[\dot{q}^2 + q^2\right] dt$ as an (L^2) inner product of $q(t)$ with an operator, Λ, acting on $q(t)$. An integration by parts immediately gives:

$$\int_0^T \left[\dot{q}^2 + q^2\right] dt = \Big(q\,,\,\Lambda(q)\Big) \quad \text{where} \quad \Lambda = -\frac{d^2}{dt^2} + 1.$$

Consider then the operator

$$\Lambda_T = -\frac{d^2}{dt^2} + 1 \quad \text{acting on } C^\infty(S^1_L, \mathbb{R}).$$

The above considerations strongly suggest that, in some sense,

$$\frac{1}{\sqrt{\det\Lambda_T}} = \frac{1}{2\sinh T/2}. \tag{6.4}$$

To make this mathematically rigorous one must first give an independent definition of $\det\Lambda_T$. Clearly the spectrum of Λ_T, with multiplicities, is

$$1 + \left(\frac{2\pi n}{T}\right)^2, \quad n = 0, \pm 1, \pm 2, \cdots$$

We cannot simply write

$$\det\Lambda_T = \prod_{n\in\mathbb{Z}} \left[1 + \left(\frac{2\pi n}{T}\right)^2\right],$$

as this infinite product is divergent. The mathematical problem we face is therefore: *how to regularize this infinite product so that (6.4) holds?*

The relevant regularization method is by the zeta function (Ray and Singer). Let's define it as follows.

Definition 6.1. Suppose $\{\lambda_n\}$ is a sequence of positive numbers such that

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

defines a holomorphic function for $\Re s$ large, and that it admits a meromrphic continuation to $\mathbb{C}$ that is regular at $s = 0$. Then

$$\prod_{n=1}^{\infty}{}^{\zeta} \lambda_n := e^{-\zeta'(0)}.$$

Example 6.2. For example, if $\forall n\ \lambda_n = n$, then $\zeta = \zeta_R$ is Riemann's zeta function. It is known that $\zeta_R'(0) = -\log(2\pi)/2$, and therefore

$$\prod_{n=1}^{\infty}{}^{\zeta} n = \sqrt{2\pi}.$$

We will need one non-trivial fact about zeta regularization:

Proposition 6.3. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two positive sequences such that*

$$\prod_{n=1}^{\infty} \frac{\lambda_n}{\mu_n} \quad \textit{converges.}$$

Then

$$\prod_{n=1}^{\infty}{}^{\zeta} \lambda_n = \left(\prod_{n=1}^{\infty}{}^{\zeta} \mu_n \right) \prod_{n=1}^{\infty} \frac{\lambda_n}{\mu_n}.$$

Using this proposition we can write:

$$\prod_{n=1}^{\infty}{}^{\zeta} \left[1 + \left(\frac{2\pi n}{T} \right)^2 \right] = \prod_{n=1}^{\infty}{}^{\zeta} \left(\frac{2\pi n}{T} \right)^2 \prod_{n=1}^{\infty} \left[1 + \left(\frac{T}{2\pi n} \right)^2 \right]. \tag{6.5}$$

On the other hand,

$$\prod_{n=1}^{\infty}{}^{\zeta} \left(\frac{2\pi n}{T} \right)^2 = e^{-\zeta'(0)}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{T} \right)^{-2s} = \left(\frac{2\pi}{T} \right)^{-2s} \zeta_R(2s).$$

Let's compute the derivative of this expression:

$$\zeta'(s) = -2\log(2\pi/T)(2\pi/T)^{-2s}\zeta_R(2s) + 2(2\pi/T)^{-2s}\,\zeta_R'(2s).$$

Evaluating at $s = 0$ (recall: $\zeta_R(0) = -1/2$

$$\zeta'(0) = \log(2\pi/T) - \log(2\pi),$$

therefore:

$$\mathop{{\prod}^{\zeta}}_{n=1}^{\infty} \left(\frac{2\pi n}{T}\right)^2 = T.$$

Going back to (6.5) one gets:

$$\mathop{{\prod}^{\zeta}}_{n=1}^{\infty} \left[1 + \left(\frac{2\pi n}{T}\right)^2\right] = T \prod_{n=1}^{\infty} \left[1 + \left(\frac{T}{2\pi n}\right)^2\right].$$

Comparing with (6.4), we conclude: *If we define the determinant of* Λ_T *using the zeta function regularization, that is, if we set*

$$\det \Lambda_T := \quad \mathop{{\prod}^{\zeta}}_{n=1}^{\infty} \left[1 + \left(\frac{2\pi n}{T}\right)^2\right]^2$$

then

$$\sqrt{\det \Lambda_T} = T \prod_{n=1}^{\infty} \left[1 + \left(\frac{T}{2\pi n}\right)^2\right]$$

and the "existence" of the path integral yields the identity

$$T \prod_{n=1}^{\infty} \left[1 + \left(\frac{T}{2\pi n}\right)^2\right] = 2\sinh(T/2).$$

This identity is true! (it is the Euler product representation of sinh.)

7. Classical Fields

We begin by considering an abstract formulation of classical field theory.

Let M be a compact, N-dimensional oriented manifold, possibly with boundary, that we interpret as "space time". Classical fields on M are sections of some fibre bundle, $F \to M$. A Lagrangian density is a (smooth) bundle map,

$$L : J^1(M,F) \to \wedge^N T^* M,$$

where $J^1(M,F)$ is the space of one-jets of sections of $F \to M$. We now recall what this space is. For each $x \in M$, consider

$$J^1(M,F)_x := \{(\ell, f)\,;\, f \in \pi^{-1}(x),\ \ell : T_x M \to T_f F\,\}$$

where ℓ is an injective linear map whose image is transverse to the tangent space to the fiber, $\pi^1(x)$. Then $J^1(M,F)$ is the fibre bundle over M

$$J^1(M,F) = \bigcup_{x \in M} J^1(M,F)_x .$$

The fundamental property of J^1 is that every section

$$\phi : M \to F$$

defines a section of $J^1(M,F)$ denoted:

$$(d\phi, \phi) : M \to J^1(M,F).$$

The Euler-Lagrange equations are the variational equations for the action

$$\mathcal{L} :\ C^\infty(M,F) \ni \phi \mapsto \int_M L(d\phi, \phi) \in \mathbb{R}$$

(Notice that in this formalism $L(d\phi,\phi)$ is a top-degree form that can therefore be integrated over M.) In the Euler-Lagrange equation,

$$\delta \mathcal{L}(\phi) = 0,$$

only variations that vanish on ∂M are considered. Let us now consider in detail a variety of examples.

Example 7.1. Take $M = [0,T]$, an interval, and $F = [0,T] \times X$ where X is some Riemannian manifold. Notice that in this case fields on M are simply curves on X, and that therefore

$$J^1(M,F) = TX, \quad \text{the tangent bundle of } X.$$

A classical Lagrangian density is of the form

$$TX \ni (x,v) \mapsto \left(\frac{1}{2}\|v\|^2 + V(x) \right) dt,$$

where $V : X \to \mathbb{R}$ is a given function and dt the line element on M. This yields classical mechanics!

Example 7.2. Let M be a compact Riemannian manifold and $F = M \times \mathbb{R}$, so that the fields are simply real-valued functions on M. In this case the space $J^1(M, F)$ is

$$J^1(M, F) = \mathbb{R} \times T^*M.$$

Fix a non-negative ingeter m and a constant λ, and consider the Lagrangian density

$$\mathbb{R} \times T^*M \ni (r, \alpha) \mapsto \left(\frac{1}{2}\|\alpha\|^2 + \lambda r^m\right) d\mu,$$

where $d\mu$ is the Riemannian volume form. The action is clearly

$$\forall \phi \in C^\infty(M) \quad \mathcal{L}(\phi) = \int_M \left(\frac{1}{2}\|d\phi\|^2 + \lambda \phi^m\right) d\mu$$

Let $\rho \in C^\infty(M)$ vanishing on ∂M, and compute

$$\frac{d}{d\epsilon}\mathcal{L}(\phi + \epsilon\rho)|_{\epsilon=0} = \int_M \left(\langle d\phi, d\rho\rangle + \lambda m \phi^{m-1}\rho\right) d\mu.$$

Doing an integration by parts one can prove that

$$\int_M \langle d\phi, d\rho\rangle \, d\mu = -\int_M \rho \Delta\phi \, d\mu$$

(where we use that ρ vanishes on ∂M), so that the Euler-Lagrange equations for this problem are

$$\Delta\phi = \lambda m \phi^{m-1}.$$

If $\lambda = 0$ we simply get: $\Delta\phi = 0$, i.e. ϕ must be a harmonic function. If $m = 1$ we get: $\Delta\phi = \lambda\phi$, i.e. ϕ must be an eigenfunction of Δ of eigenvalue λ.

Example 7.3. Let $F = T^*M$, so that fields are one-forms on M.

$$J^1(M, T^*M) = \ldots$$

Suppose M carries a Riemannian metric, and consider the Lagrangian density

8. Symmetry in Classical Mechanics

8.1. **The Noether principle.** The Noether principle states that if a classical dynamical system possesses a Lie group of symmetries, then the system has "conserved quantities", that is, functions on phase space that are constant along the orbits of the system. Thus, in the Hamiltonian formulation, if H is the Hamiltonian and F a function on phase space, F is a conserved quantity iff $\{H, F\} = 0$. Conserved quantities should be thought of as "partially integrating" the equations of motion, since the orbits of the system are contained in the joint level sets of the conserved quantities. For this reason conserved quantities are traditionally referred to as "first integrals" of motion.

8.1.1. *Hamiltonian formulation.* From the Hamiltonian perspective, a natural setting for the discussion of Noether's principle is a symplectic manifold, (X, ω), on which a Lie group, G, acts by automorphisms:

$$G \times X \ni (g, x) \mapsto g \cdot x =: \rho_g(x) \in X.$$

The "automorphism" condition on the action is that $\forall g \in G \ \rho_g^* \omega = \omega$. As with any action, one gets a map at the Lie algebra level: For each $A \in \mathfrak{g}$ let $A^\#$ be the vector field on X defined by:

$$\forall x \in X \qquad A^\#_x = \frac{d}{dt} \exp(tA) \cdot x|_{t=0}.$$

The mapping $A \mapsto A^\#$ is a Lie algebra morphism:

$$\forall A,\ B \in \mathfrak{g} \qquad [A\,,\,B]^\# = [A^\#\,,\,B^\#]$$

where the bracket on the right is the commutator of vector fields. Let us investigate the infinitesimal version of $\rho_g^* \omega = \omega$, which is clearly

$$0 = \frac{d}{dt}\exp(tA)^* \omega|_{t=0} = \mathcal{L}_{A^\#}\omega.$$

By the infinitesimal homotopy identity,

$$\mathcal{L}_{A^\#}\omega = d(A^\# \rfloor \omega) + d\omega \rfloor A^\#;$$

since $d\omega = 0$ this implies: $d(A^\# \rfloor \omega) = 0$. Therefore, at least locally on X, for each $A \in \mathfrak{g}$ there exists a function Φ_A such that

$$A^\# \rfloor \omega = d\Phi_A,$$

that is, $A^\#$ *is a (locally) Hamiltonian vector field.* If $H \in C^\infty(X)$ is a Hamiltonian invariant under the action of G, then $\forall A \in \mathfrak{g}$

$$0 = \mathcal{L}_{A^\#} H = \{\Phi_A\,,\, H\}.$$

This is the Hamiltonian Noether principle: The existence of a Lie group of symmetries of (X, ω, H) implies the existence of (locally-defined) conserved quantities.

8.2. **The moment map.** In many cases of interest the conserved quantities associated to an action of a Lie group G on (X, ω) exist globally.

Definition 8.1. Let a connected Lie group G act on (X, ω) by automorphisms. We say the action is *Hamiltonian* iff there is a Lie algebra morphism, $\mathfrak{g} \to C^\infty(X)$, denoted

$$\mathfrak{g} \ni A \mapsto \Phi_A \in C^\infty(X),$$

such that, $\forall A \in \mathfrak{g}$, $A^\#$ is the Hamilton vector field of Φ_A, i.e. $d\Phi_A = \omega \rfloor A^\#$.

Remark 8.2. The condition that $A \mapsto \Phi_A$ be a Lie algebra morphism is:

$$\forall A,\ B \in \mathfrak{g} \qquad \Phi_{[A,B]} = \{\Phi_A\,,\ \Phi_B\}$$

There is a beautiful way of reformulating the map $A \mapsto \Phi_A$ as a map between finite-dimensional spaces:

Definition 8.3. Assume G acts on (X, ω) in a Hamiltonian fashion. The associated moment map, $\Phi : X \to \mathfrak{g}^*$, is then defined by the formula:

$$\forall x \in X \quad \Phi(x) \text{ is the linear map: } \begin{array}{ccc} \mathfrak{g} & \to & \mathbb{R} \\ A & \mapsto & \Phi_A(x) \end{array}$$

Example 8.4. Let $X = \mathbb{R}^{2n}$ with its standard symplectic form, and let $G = \mathrm{Sp}(n)$. Lemma 3.3 precisely says that the map

$$\mathrm{sp}(n) \ni A \mapsto \Phi_A \in S^2\mathbb{R}^{2n}, \quad \Phi_A(v) = \frac{1}{2}\omega(v\,,\ Av)$$

defines a moment map for the natural action of $\mathrm{Sp}(n)$ on $\mathbb{R}^{2n}$.

The previous example generates many others, by virtue of the following

Lemma 8.5. *Let G act in a Hamiltonian fashion on (X, ω), with moment map $\Phi : X \to \mathfrak{g}^*$. Let $K \subset G$ be a closed subgroup. Then the induced action of K on X is Hamiltonian, with moment map equal to the composition*

$$J : X \xrightarrow{\Phi} \mathfrak{g}^* \to \mathfrak{k}^*$$

where the second arrow is the dual of the inclusion $\mathfrak{k} \hookrightarrow \mathfrak{g}$.

The theory of the moment map is long and beautiful; see [G-S *Symplectic techniques in Physics*], for example. Here we will only state some very elementary results.

Lemma 8.6. *A moment map $\Phi : X \to \mathfrak{g}^*$, defined as above, is equivariant: it intertwines the given action of G on X and the coadjoint action of G on $\mathfrak{g}^*$.*

Lemma 8.7. *Let $\Phi : X \to \mathfrak{g}^*$ be a moment map, and let $x \in X$. Then the differential, $d\Phi_x : T_xX \to \mathfrak{g}^*$, is the transpose of the map which is the composition*

$$\mathfrak{g} \to T_xX \to T_x^*X,$$

where the first arrow is: $A \mapsto A_x^\#$ and the second is contraction with ω.

Proof. Let $v \in T_xX$ and $A \in \mathfrak{g}$. What we need to prove is that

$$\langle d\Phi_x(v), A\rangle = \omega(v, A_x^\#).$$

Let $x(t)$ be a curve on X such that $x(0) = x$ and $\dot{x}(0) = v$. Then, since the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is bilinear,

$$\langle d\Phi_x(v), A\rangle = \frac{d}{dt}\langle \Phi(x(t)), A\rangle|_{t=0}.$$

But, by definition, $\langle \Phi(x(t)), A\rangle = \Phi_A(x(t))$, therefore

$$\langle d\Phi_x(v), A\rangle = d(\Phi_A)_x(v).$$

Finally, simply recall that $A^\#$ is the Hamilton vector field of Φ_A, which means precisely that

$$d(\Phi_A)_x(v) = \omega(v, A_x^\#).$$

□

The previous lemma has some important consequences. Let $G \cdot x$ denote the orbit of G through x:

$$G \cdot x = \{\, g \cdot x \ ; \ g \in G \,\}.$$

This is an (possibly immersed, if G is non-compact) submanifold of X. Let G_x the isotropy subgroup of x:

$$G_x = \{g \in G \ ; \ g \cdot x = x\,\}.$$

This is a Lie subgroup; we denote its Lie algebra by $\mathfrak{g}_x$. It should be clear that

$$\mathfrak{g}_x = \{\, A \in \mathfrak{g} \ ; \ A_x^\# = 0\,\}.$$

As manifolds, $G \cdot x \cong G/G_x$.

Corollary 8.8. 1. *The kernel of $d\Phi_x$ is the symplectic perpendicular to the G orbit at x, that is*

$$\ker(d\Phi_x) = T_x(G \cdot x)^\perp.$$

2. *The image of* $d\Phi_x$ *is the annihilator of* $\mathfrak{g}_x$, *i.e.*

$$\text{Image } d\Phi_x = \{\xi \in \mathfrak{g}^* \ ; \ \forall A \in \mathfrak{g}_x \langle \xi, A\rangle = 0\,\}.$$

8.3. **Symplectic quotients.** Whenever one has some kind of G space, the question arises: how does one define the appropriate quotient? The simplest definition, in the Hamiltonian setting, turns out to be the following:

Definition 8.9. Let G act on (X, ω), with moment map Φ. The symplectic quotient of X by G is defined as:

$$X/\!\!/G := \Phi^{-1}(0)/G,$$

where the right-hand side denotes the space of G orbits in $\Phi^{-1}(0)$.

As we will see, if 0 *is a regular value of* Φ, $X/\!\!/G$ *inherits a natural symplectic structure from that of* X. The first step in this direction is to understand the structure of

$$Z := \Phi^{-1}(0).$$

For the remainder of this section we will assume: 0 is a regular value of Φ.

Lemma 8.10. *Z is a co-isotropic submanifold of X, that is*

$$\forall x \in Z \qquad \Big(T_x Z\Big)^{\perp} \subset T_x Z,$$

where $\perp$ denotes the symplectic orthogonal:

$$\Big(T_x Z\Big)^{\perp} = \{\, v \in T_x X \ ; \ \forall w \in T_x Z \quad \omega(v, w) = 0\,\}.$$

Proof. This is just because $T_x Z = \ker d\Phi_x$, Corollary 8.8 and the fact that Z is G-invariant by the equivariance of Φ. $\square$

Notice that, again by Corollary 8.8, $\forall x \in Z$ the isotropy subalgebra, $\mathfrak{g}_x$, is zero. Therefore the isotropy subgroup, G_x, is discrete. We will next assume that the G_x are trivial:

Proposition 8.11. *Let $\iota : Z \hookrightarrow X$ be inclusion. Assume that G acts freely on Z, i.e. $G_x = \{1\}$ $\forall x \in Z$. Let $\pi : Z \to Z/G$. Then there exists a unique two-form, Ω, on Z/G such that*

$$\iota^*\omega = \pi^*\Omega, \qquad \begin{array}{ccc} Z & \overset{\iota}{\hookrightarrow} & X \\ \downarrow \pi & & \\ Z/G & & \end{array}$$

Moreover, Ω is a symplectic form on $Z/G = X/\!\!/G$.

Proof. Notice first of all that the assumptions imply that Z/G is a manifold. We will reason locally. Let $U \subset Z/G$ be a coordinate chart, with coordinates $x_1, \dots x_m$. Denote also by x_j the function: $\pi^* x_j$. Let $y_1, \dots y_k$ be functions defined in $V \subset \pi^{-1}(U)$ such that $x_1, \dots x_m, y_1, \dots y_k$ are coordinates on V. Then we can write, with obvious limits of summation:

$$\iota^*\omega = \sum f_{ij}\ dx_i \wedge dx_j + \sum g_{ij}\ dx_i \wedge dy_j + \sum h_{ij}\ dy_i \wedge dy_j.$$

On V the fibers of π are defined by the equations: $x =$ const. Since the fibers are coisotropic, $\omega \rfloor \partial/\partial y_j = 0$, which implies $g_{ij} = 0,\ h_{ij} = 0$, for all relevant indices. On the other hand

$$0 = d\iota^*\omega = \sum \frac{\partial f_{ij}}{\partial y_l}\ dy_l \wedge dx_i \wedge dx_j + \sum \frac{\partial f_{ij}}{\partial x_l}\ dx_l \wedge dx_i \wedge dx_j.$$

It follows that $\frac{\partial f_{ij}}{\partial y_l} = 0$ for all indices, i.e. the f_{ij} are functions on U; in addition the previous equation shows that

$$\Omega = \sum f_{ij}\ dx_i \wedge dx_j$$

is a closed form on U. Uniqueness is obvious. □

8.4. **Generalizations of the quotient procedure.**

8.5. **An infinite-dimensional example.** We discuss, in this subsection, an example that will be generalized in a subsequent section on the Yang-Mills theory.

Let Σ be a compact oriented surface, and let

$$X = \{\text{ smooth real-valued one-forms on } \Sigma\,\}.$$

X is in fact a symplectic vector space, with symplectic form, Ω, defined by:

$$\forall \alpha,\ \beta \in X \qquad \Omega(\alpha\,,\,\beta) = \int_\Sigma \alpha \wedge \beta.$$

It is easy to see that Ω is non-degenerate: if one puts a Riemannian structure on Σ and $J : T\Sigma \to T\Sigma$ is "rotation by $\pi/2$", then

$$\Omega(\alpha, J^*\alpha) = \|\alpha\|^2.$$

Furthermore, Ω is (formally) closed since it is constant (with respect to the translations on X).

There is an infinite-dimensional Lie group acting on X, namely

$$\mathcal{G} := C^\infty(\Sigma, S^1),$$

the group structure being pointwise multiplication. It will be useful to observe that any function, $h \in C^\infty(\Sigma, \mathbb{R})$, defines an element of $\mathcal{G}$ by the formula: $g = e^{ih}$.

Not every element of $\mathcal{G}$, however, is of this form: the logarithm of $g : \Sigma \to S^1$ is only defined mod $\mathbb{Z}$.

The action of $\mathcal{G}$ on X is:

$$\forall g \in \mathcal{G},\ \alpha \in X \qquad g \cdot \alpha = \alpha + \frac{1}{i} g^{-1} dg.$$

We are thinking here that S^1 is the group of complex numbers of modulus one. Locally on Σ g has a logarithm: $g = e^{ih}$, and therefore locally $g \cdot \alpha = \alpha + dh$.

Is this action Hamiltonian? To begin to answer this question, we must understand what the Lie algebra of $\mathcal{G}$, Lie$\mathcal{G}$, is. Thus let g_t be a one-parameter subgroup of $\mathcal{G}$. g_t is in fact a smooth function

$$(-\epsilon, \epsilon) \times \Sigma \to S^1$$

that satisfies:

$$\forall x \in \Sigma \qquad g_t(x) g_s(x) = g_{t+s}(x) \quad \text{and } g_0(x) = 1.$$

An element of Lie$\mathcal{G}$ is the derivative of such a function, with respect to $t \in (-\epsilon, \epsilon)$, at $t = 0$. The above group law shows that, $\forall x \in \Sigma$, $t \mapsto g_t(x)$ is a one-parameter subgroup of S^1, and therefore there exists a number, $\phi(x) \in \mathbb{R}$, such that $g_t(x) = e^{it\phi(x)}$. Smoothness of g_t implies that $x \mapsto \phi(x)$ is a smooth function.

9. Co-adjoint orbits.

9.1. **Their symplectic form.** Let G denote a connected Lie group with Lie algebra $\mathfrak{g}$. Recall that G acts on $\mathfrak{g}$ by the so-called adjoint representation, defined as follows. Firstly, G acts on itself by conjugation: $g \cdot x = gxg^{-1}$. The identity, $1 \in G$, is a fixed point for this action. This fact makes the following definition possible:

Definition 9.1. If $g \in G$, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of the map

$$\begin{array}{ccc} G & \to & G \\ x & \mapsto & gxg^{-1}. \end{array}$$

Notice that Ad_g is a linear map, for each g. By the chain rule, $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$.

Consider the dual of the vector space $\mathfrak{g}$, which we denote by $\mathfrak{g}^*$. The adjoint action of G on $\mathfrak{g}$ can be "dualized" to yield an action of G on $\mathfrak{g}^*$, the co-adjoint representation:

Definition 9.2. If $g \in G$, $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined as the transpose of $\mathrm{Ad}_{g^{-1}}$, that is:

$$\forall A \in \mathfrak{g}, \quad \xi \in \mathfrak{g}^* \qquad \langle \mathrm{Ad}^*_g(\xi), A\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}(A)}\rangle,$$

where the brackets denote the natural pairing, $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

The reason for using the inverse of g in this definition is to preserve the property: $\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h = \mathrm{Ad}^*_{gh}$.

Remark: In many cases, but not always, there exists a non-degenerate inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G. The linear isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ associated to such an inner product intertwines the Ad and the Ad^* actions, and therefore it can be used to identify adjoint and coadjoint orbits. The identification is such that the natural pairing above simply becomes the inner product, $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Exercise 9.1.1. Define $\langle \cdot, \cdot \rangle : \mathfrak{gl}(n) \times \mathfrak{gl}(n) \to \mathbb{R}$ by:

$$\langle A, B\rangle = \frac{1}{2}\,\mathrm{Tr}(AB).$$

Let G be a matrix group, and consider the restriction, $\langle , \rangle_{\mathfrak{g}}$, of the above, to $\mathfrak{g} \times \mathfrak{g}$.

(a) Show that $\langle , \rangle_{\mathfrak{g}}$ is Ad-invariant.
(b) Show that if G is the orthogonal group, $\langle , \rangle_{\mathfrak{g}}$ is non-degenerate.
(c) Identify the unitary group $\mathrm{U}(n)$ with a subgroup, G, of $\mathrm{GL}(2n, \mathbb{R})$ and show that $\langle , \rangle_{\mathfrak{g}}$ is non-degenerate. Write $\langle , \rangle_{\mathfrak{g}}$ in complex notation, i.e. acting on pairs of skew-Hermitian matrices.

Co-adjoint orbits are simply orbits, $\mathcal{O} \subset \mathfrak{g}^*$, of G under the Ad^* action. The extraordinary feature of coadjoint orbits is that *they all carry a natural symplectic structure:*

Theorem 9.3. *For any connected Lie group G, every coadjoint orbit, $\mathcal{O} \subset \mathfrak{g}^*$, possesses a natural G-invariant symplectic structure*

The remainder of this section is devoted to the proof of this fact. We begin by defining a two-form on $\mathcal{O}$, which we'll then prove is a symplectic structure. Let $\xi \in \mathcal{O}$. If $A \in \mathfrak{g}$, let

$$A^{\#}_{\xi} := \frac{d}{dt}\, \mathrm{Ad}^*_{\exp(tA)}(\xi)|_{t=0}. \tag{9.1}$$

Since the action of G on $\mathcal{O}$ is transitive (by definition!), the map

$$\begin{array}{ccc} \mathfrak{g} & \to & T_x\mathcal{O} \\ A & \mapsto & A^{\#}_{\xi} \end{array}$$

induces a natural isomorphism

$$T_\xi\mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\xi, \tag{9.2}$$

where $\mathfrak{g}_\xi = \{\, A \in \mathfrak{g} \ ; \ A^{\#}_{\xi} = 0 \,\}$. This is the Lie algebra of the isotropy subgroup of ξ:

$$G_\xi = \{\, g \in G \ ; \ \mathrm{Ad}^*_g(\xi) = \xi \,\}.$$

We first need a description of this Lie algebra:

Lemma 9.4. $\mathfrak{g}_\xi = \{\, A \in \mathfrak{g} \ ; \ \forall B \in \mathfrak{g} \quad \langle \xi, [A,B]\rangle = 0 \,\}$.

Proof. $A \in \mathfrak{g}_\xi$ if and only if for all t $\mathrm{Ad}^*_{\exp(tA)}(\xi) = \xi$. This is equivalent to: for all t and all $B \in \mathfrak{g}$

$$0 = \langle \mathrm{Ad}^*_{\exp(tA)}(\xi), B\rangle = \langle \xi, \mathrm{Ad}_{\exp(-tA)}(B)\rangle.$$

Now take d/dt and evaluate at $t = 0$. □

Consider now the skew-symmetric bilinear form

$$\begin{array}{rccc} \tilde{\sigma}_\xi : & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{R} \\ & (A,B) & \mapsto & \langle \xi, [A,B]\rangle. \end{array} \tag{9.3}$$

The previous Lemma is saying precisely that the kernel of this form is $\mathfrak{g}_\xi$. Therefore, via the identification (9.2), (9.3) defines a *non-degenerate* skew-symmetric bilinear form on $T_\xi\mathcal{O}$. Specifically, this form is:

$$\begin{array}{rccc} \sigma_\xi : & T_\xi\mathcal{O} \times T_\xi\mathcal{O} & \to & \mathbb{R} \\ & (A^{\#}_{\xi}, B^{\#}_{\xi}) & \mapsto & \langle \xi, [A,B]\rangle. \end{array} \tag{9.4}$$

It should be easy to accept that, as ξ varies in $\mathcal{O}$, the above defines a smooth, G-invariant, two-form on $\mathcal{O}$.

Definition 9.5. σ called the Kostant-Kirilov (symplectic) form.

To prove that σ is symplectic all we have left to prove is: $d\sigma = 0$. Let $A,\ B,\ C \in \mathfrak{g}$ and consider the corresponding vector fields, $A^{\#},\ B^{\#},\ C^{\#}$ on $\mathcal{O}$, defined by (9.1). We will use the identity, valid for any two-form σ and any three vector fields:

$$3(d\sigma)(A^{\#}, B^{\#}, C^{\#}) = \tag{9.5}$$

$$A^{\#}(\sigma(B^{\#}, C^{\#})) + B^{\#}(\sigma(C^{\#}, A^{\#})) + C^{\#}(\sigma(A^{\#}, B^{\#}))$$

$$-\sigma([A^{\#}, B^{\#}], C^{\#}) - \sigma([B^{\#}, C^{\#}], A^{\#}) - \sigma([C^{\#}, A^{\#}], B^{\#}).$$

Lemma 9.6. *$A^{\#}(\sigma(B^{\#}, C^{\#})) = \sigma([B^{\#}, C^{\#}], A^{\#})$.*

Proof. By definition of σ, $\sigma(B^{\#}, C^{\#})$ is the function:

$$\mathcal{O} \ni \xi \mapsto \langle \xi, [B, C] \rangle.$$

Thus the left-hand side of the identity to be proven is the derivative at $t = 0$ of:

$$\langle \mathrm{Ad}^*_{\exp(tA)}(\xi), [B, C] \rangle = \langle \xi, \mathrm{Ad}_{\exp(-tA)}([B, C]) \rangle.$$

However,

$$\frac{d}{dt}\mathrm{Ad}_{\exp(-tA)}([B, C])|_{t=0} = -[A\,,\ [B, C]].$$

Therefore

$$A^{\#}(\sigma(B^{\#}, C^{\#}))\,(\xi) = -\langle \xi, [A\,,\ [B, C]] \rangle = \langle \xi, [[B, C]\,,\ A] \rangle = \sigma([B^{\#}, C^{\#}], A^{\#})(\xi).$$

$\square$

It follows from this Lemma that $(d\sigma)(A^{\#}, B^{\#}, C^{\#}) = 0$. But since, for a given $\phi \in \mathcal{O}$, any tangent vector in $T_\phi \mathcal{O}$ is of the form $A^{\#}_\phi$ for some $A \in \mathfrak{g}$, it follows that $d\sigma = 0$.

9.2. **Their moment map.** Let $\mathcal{O}$ be a coadjoint orbit of a Lie group, G. The vector fields $A^{\#}$, $A \in \mathfrak{g}$, are the infinitesimal generators of the G action on $\mathcal{O}$. Since -as we accepted- the action preserves the Kostant-Krilov symplectic form, the $A^{\#}$ are locally Hamiltonian. In fact, these fields are always globally Hamitlonian:

Lemma 9.7. *Let $A \in \mathfrak{g}$, and consider the function, $\ell_A : \mathcal{O} \to \mathbb{R}$ defined by:*

$$\forall \phi \in \mathcal{O} \qquad \ell_A(\phi) = \langle \phi, A \rangle.$$

(i.e. ℓ_A is the restriction to $\mathcal{O}$ of the linear function on $\mathfrak{g}^*$ defined by A.) Then the Hamilton vector field of ℓ, with respect to the Kostant-Kirilov symplectic form on $\mathcal{O}$, is precisely $A^\#$.

Proof. If $B \in \mathfrak{g}$ and $\phi \in \mathcal{O}$, by definition,

$$\sigma(A^\#, B^\#)(\phi) = \langle \phi, [A, B] \rangle.$$

Therefore, to prove the Lemma it suffices to show that $d(\ell_A)_\phi(B^\#_\phi) = \langle \phi, [A, B] \rangle$. But by definition of $B^\#$,

$$d(\ell_A)_\phi(B^\#_\phi) = \frac{d}{dt}\ell_A(\mathrm{Ad}^*_{\exp(tB)}(\phi))|_{t=0} = \frac{d}{dt}\langle \mathrm{Ad}^*_{\exp(tB)}(\phi), A\rangle|_{t=0} = -\langle \phi, [B, A]\rangle.$$

$\square$

Corollary 9.8. *The natural action of G on $\mathcal{O}$ is Hamiltonian, with moment map $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ the inclusion.*

Notice, by the way, that the obvious equivariance of the moment map implies that

$$\ell_{[A,B]} = \{\ell_A\,,\, \ell_B\}.$$

9.3. **Co-adjoint orbits of $U(n)$: a problem set.** Let $\alpha = (\alpha_1, \dots, \alpha_n)$ where $\alpha_1 \geq \cdots \geq \alpha_n$ are real numbers, and consider

$$X_\alpha = \{\, n \times n \text{ Hermitian matrices, } L,\ \mathrm{spec}(L) = \alpha \,\}.$$

So $L \in X_\alpha$ iff the eigenvalues of L are the α_j. Notice that $U(n)$ acts on X_α, by conjugation. Throughout, identify $u(n)$ with its dual using the results from Exercise 9.1.1.

Exercise 9.3.1. Identify X_α with a co-adjoint orbit of $U(n)$. Conversely, show that every co-adjoint orbit of $U(n)$ can be identified with X_α for some α.

Exercise 9.3.2. What is the dimension of X_α? Your answer will involve the multiplicities of the eigenvalues α. Show that if α and β are two decreasing sequences of eigenvalues with the same multiplicities, there exists a covariant diffeomorphism $F : X_\alpha \to X_\beta$. HINT: Find the isotropy subgroup of diagonal elements in X_α.

Exercise 9.3.3. Arguing that $T_L X_\alpha = \{\, [L, A]\ ;\ A \in u(n) \,\}$, explicitly describe the Kostant-Kirilov symplectic form on X_α. Then show that an equivariant diffeomorphism, $F : X_\alpha \to X_\beta$, is a symplectomorphism iff it is the identity.

Exercise 9.3.4. Let $T \subset U(n)$ be the subgroup of diagonal unitary matrices, and consider X_α as a Hamiltonian T space.

(a) Give a concrete description of the moment map, $J : X_\alpha \to \mathfrak{t}^*$, in linear algebra language.

(b) A Theorem of Kostant states that the image of J is the convex hull of $J(X_\alpha^T)$, where

$$X_\alpha^T = \{\, x \in X_\alpha \; ; \; \forall t \in T \; txt^{-1} = x \,\}.$$

Sketch the possible images of J for $n = 2, 3$, identifying $\mathfrak{t}^*$ with $\mathbb{R}^n$. For $n = 3$, show that the image is contained in plane $\sum_{j=1}^{3} x_j = \sum_{j=1}^{3} \alpha_j$, and sketch the image on this plane.